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New strings for old Veneziano amplitudes II. Group-theoretic treatment

A.L. Kholodenko*

375 H.L. Hunter Laboratories, Clemson University, Clemson, SC 29634-0973, USA

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Abstract

In this part of our four parts work we use theory of polynomial invariants of finite pseudo-reflection groups in order to reconstruct both the Veneziano and Veneziano-like (tachyon-free) amplitudes and the generating function reproducing these amplitudes. We demonstrate that such generating function and amplitudes associated with it can be recovered with help of *finite dimensional exactly solvable* $N = 2$ supersymmetric quantum mechanical model known earlier from works of Witten, Stone and others. Using the Lefschetz isomorphism theorem we replace traditional supersymmetric calculations by the group-theoretic thus solving the Veneziano model exactly using standard methods of representation theory. Mathematical correctness of our arguments relies on important theorems by Shepard and Todd, Serre and Solomon proven respectively in the early 50s and 60s and documented in the monograph by Bourbaki. Based on these theorems, we explain why the developed formalism leaves all known results of conformal field theories unchanged. We also explain why these theorems impose stringent requirements connecting analytical properties of scattering amplitudes with symmetries of space-time in which such amplitudes act.

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* Tel.: +1 8646565016; fax: +1 8646566613.
E-mail address: string@clemson.edu.

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1. Introduction

1.1. Motivation

In our earlier work, Ref. [1], which will be called Part I, while discussing analytical properties of the Veneziano and Veneziano-like amplitudes we noticed that the Veneziano condition for the four-particle amplitude is given by

$$\alpha(s) + \alpha(t) + \alpha(u) = -1, \tag{1.1}$$

where $\alpha(s)$, $\alpha(t)$, $\alpha(u) \in \mathbf{Z}$. This result can be rewritten in more general and mathematically suggestive form. To this purpose, following Ref. [2], we would like to consider additional homogenous equation of the type

$$\alpha(s)m + \alpha(t)n + \alpha(u)l + k \cdot 1 = 0, \tag{1.2}$$

with m, n, l, k being some integers. By adding this equation to Eq. (1.1) we obtain,

$$\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} = \tilde{k}, \tag{1.3a}$$

so that we formally obtain,

$$n_1 + n_2 + n_3 = \hat{N}, \tag{1.3b}$$

where all entries *by design* are nonnegative integers. For the multiparticle case this equation should be replaced by

$$n_0 + \cdots + n_k = N, \tag{1.4}$$

so that combinatorially the task lies in finding all nonnegative integer combinations of $n_0, \ldots, n_k$ producing Eq. (1.4). It should be noted that such a task makes sense as long as N is assigned. But the actual value of N is *not fixed* and, hence, can be chosen quite arbitrarily.

Remark 1.1. In view of the results of Part I, one can argue that the value of N should coincide with the exponent of the Fermat (hyper)surface. This observation is superficial, however, in view of Eq. (3.29) of Part I. Indeed, in this Section we are talking about the mathematical statements *before* the bracket operation $\langle \ldots \rangle$ defined in Part I is applied. This means that we shall be working mainly with precursors of the period integrals in the projective space discussed in some detail in Section 3 of Part I. Evidently, this makes sense only if, in contrast with traditional string-theoretic treatments, we interpret the Veneziano amplitudes as periods of the Fermat (hyper)surfaces.

Remark 1.2. Eq. (1.1) is a simple statement about the energy–momentum conservation. Although the numerical entries in this equation can be changed as we just have explained to make them more suitable for theoretical treatments, the actual physical values can be reobtained subsequently by the appropriate coordinate shift. Such a procedure is not applicable to amplitudes in conformal field theories (CFT) where the periodic (antiperiodic, etc.) boundary conditions cause energy and momenta to become a *quasi-energy* and a *quasi momenta* as is well known from the solid state physics. This fact was noticed already in Part I where Eq. (3.22) used in CFT replaces the standard Veneziano condition, e.g. Eq. (1.2).

This arbitrariness of choosing N represents a kind of gauge freedom in physics terminology. As in other gauge theories, we can fix the gauge by using some physical considerations. These include, for example, an observation made in Part I that the four particle amplitude is zero if any two entries into Eq. (1.1) (or, which is the same, into Eq. (1.3b)) are the same. This fact prompts us to arrange the entries in Eq. (1.3b) in accordance with their magnitude, i.e. $n_1 \geq n_2 \geq n_3$. More generally, in view of Eq. (1.4), we can write: $n_0 \geq n_1 \geq \cdots \geq n_k \geq 1$.[1] Provided that Eq. (1.4) holds, we shall call such a sequence a *partition* and shall denote it as $n \equiv (n_0, \ldots, n_k)$. If n is partition of N, then we shall write $n \vdash N$. It is well known [3,4] that there is one-to-one correspondence between the Young diagrams and partitions. We would like to use this fact in order to design a new partition function capable of reproducing the Veneziano (and Veneziano-like) amplitudes. Clearly, such a partition function should also make physical sense. This is the primary goal of our paper. In this section we would like to provide some convincing qualitative arguments that such a goal can indeed be achieved. The rest of the paper provides more rigorous mathematical results supporting our claim.[2]

We begin with observation (taken from our earlier study of the Witten–Kontsevich model, Ref. [7]) that there is one-to-one correspondence between the Young tableaux and directed random walks. Let us recall details of this correspondence now. To this purpose we need to consider a square lattice and to place on it the Young diagram associated with some specific partition which belongs to n. To do so, let us choose some $\tilde{n} \times \tilde{m}$ rectangle[3] so that the Young diagram occupies the left part of this rectangle. We choose the upper left vertex of the rectangle as the origin of the xy coordinate system whose y axis (south direction) is directed downwards and x axis is directed eastwards. Then, the southeast boundary of the Young diagram can be interpreted as directed (that is without self intersections) random walk which begins at $(0, -\tilde{m})$ and ends at $(\tilde{n}, 0)$. Clearly, such a walk completely determines the diagram. The walk can be described by a sequence of 0's and 1's, say, 0 for the x-step move and 1 for the y-step move. The totality $\mathcal{N}$ of Young diagrams which can be placed in the rectangle is in one-to-one correspondence with the number of arrangements of 0's and 1's whose total number is $\tilde{m} + \tilde{n}$. The logarithm of the number $\mathcal{N}$ of possible combinations of 0's and 1's is just the entropy associated with the Fermi statistic (or, equivalently, the entropy of mixing for the binary mixture) used in physics literature. The number $\mathcal{N}$ is given by $\mathcal{N} = (m+n)!/m!n!$.[4] It can be represented in two equivalent ways

$$\frac{(m+n)!}{m!n!} = \frac{(n+1)(n+2)\cdots(n+m)}{m!} \equiv \begin{pmatrix} n+m \\ m \end{pmatrix}$$
$$= \frac{(m+1)(m+2)\cdots(n+m)}{n!} \equiv \begin{pmatrix} m+n \\ n \end{pmatrix}. \tag{1.5}$$

In Part I, Eqs. (1.21)–(1.23) explain how $\mathcal{N}$ is entering the Veneziano amplitude. Additional physical significance of this number in connection with the Veneziano amplitude and its partition function is developed in Ref. [6] and in Part III. For such a development it is absolutely essential

[1] The last inequality: $n_k \geq 1$, is chosen only for the sake of comparison with the existing literature conventions, e.g. see Ref. [3].

[2] We would like to warn our readers that, actually, there are several interrelated formulations of this partition function. Ref. [6] and Part III provide some examples of such formulations. Part IV provides additional requirements aimed to connect our formulations with the experimental data.

[3] The parameters $\tilde{n}$ and $\tilde{m}$ will be specified shortly below.

[4] We have suppressed the tildas for n and m in this expression since these parameters are going to be redefined below anyway.

that the number $\mathcal{N}$ is integer *for all nonnegative m's and n's*[5] *and can be presented in two ways*. In Part I we noticed that $\mathcal{N}$ can be interpreted as the total number of points with integer coordinates enclosed by the dilated (with dilation coefficient n) m-dimensional simplex $n\Delta_m$ whose vertices are located at the nodes of $\mathbf{Z}^m$. This observation is crucial for development of our formalism, especially in Part III.

Let now $p(N; k, m)$ be the number of partitions of N into $\leq k$ nonnegative parts, each not larger than m. Consider the generating function of the following type:

$$\mathcal{F}(k, m|q) = \sum_{N=0}^{S} p(N; k, m) q^N, \tag{1.6}$$

where the upper limit S will be determined shortly below. It is shown in Refs. [3–5,8] that $\mathcal{F}(k, m|q) = \begin{bmatrix} k+m \\ m \end{bmatrix}_q \equiv \begin{bmatrix} k+m \\ k \end{bmatrix}_q$ where, for instance, $\begin{bmatrix} k+m \\ m \end{bmatrix}_{q=1} = \begin{pmatrix} k+m \\ m \end{pmatrix}$.[6] It should be clear from this result that the expression $\begin{bmatrix} k+m \\ m \end{bmatrix}_q$ is a q-analog of the binomial coefficient $\begin{pmatrix} k+m \\ m \end{pmatrix}$. In literature [3–5,8] this q-analog is known as the *Gaussian* coefficient. Explicitly,

$$\begin{bmatrix} k \\ m \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}. \tag{1.7}$$

From this definition, it should be intuitively clear that the sum defining the generating function $\mathcal{F}(k, m|q)$ in Eq. (1.6) should have only *finite* number of terms. Eq. (1.7) allows easy determination of the upper limit S in the sum, Eq. (1.6). It is given by km. This is just the area of the $k \times m$ rectangle. Evidently, in view of the definition of $p(N; k, m)$, the number $m = N - k$. Using this fact, Eq. (1.6) can be rewritten as: $\mathcal{F}(N, N-k|q) = \begin{bmatrix} N \\ k \end{bmatrix}_q$. This expression happens to be the Poincaré polynomial for the complex Grassmannian $\mathrm{Gr}(m, k)$. This can be found on page 292 of the famous book by Bott and Tu, Ref. [9].[7] From this point of view the numerical coefficients, i.e. $p(N; k, m)$, in the q expansion of Eq. (1.6) should be interpreted as the Betti numbers of this Grassmannian. They can be determined recursively using the following property of the Gaussian

[5] That this should be the case can be seen by noticing that using symmetry considerations we can always write $(m+n)! = m!n!C(m, n)$. The constant $C(m, n)$ represents all permutations between sets m and n. It should be a positive integer since the l.h.s.in the above equation *is* a positive integer.

[6] On page 15 of the book by Stanley, Ref. [4], one can find that the number of solutions $N(n, k)$ in *positive* integers to $y_1 + \cdots + y_k = n + k$ is given by $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$ while the number of solutions in *nonnegative* integers to $x_1 + \cdots + x_k = n$ is $\begin{pmatrix} n+k \\ k \end{pmatrix}$. Careful reading of page 15 indicates however that the last number refers to solution in nonnegative integers of the equation $x_0 + \cdots + x_k = n$. We have used this fact in Part I, e.g. see Eq. (1.21).

[7] To make a comparison it is sufficient to replace parameters t^2 and n in Bott and Tu book by q and N.

coefficients [4, p. 26]

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n \\ k+1 \end{bmatrix}_q + q^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q , \tag{1.8}$$

and taking into account that $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$. To demonstrate that $\mathcal{F}(N, N-k|q)$ is indeed the Poincaré polynomial for the Grassmannian we would like to use some results from the number theory. For readers unfamiliar with number theory a concise summary of relevant results can be found, for instance, in Appendix A of our earlier work, Ref. [10]. Given this, let q be some prime and consider the finite field $\mathbf{F}_q$ of q elements. Consider next the field extension. This is effectively accomplished by constructing an N dimensional vector space via prescription:

$$\mathbf{F}_q^N = \{\alpha_1, \ldots, \alpha_N\} : \alpha_i \in \mathbf{F}_q. \tag{1.9}$$

Any number which belongs to this new (extended) number field is expandable in terms of the basis "vectors" just specified. It can be shown [2,8], that the number of k-dimensional subspaces of the vector space $\mathbf{F}_q^N$ is given exactly by $\mathcal{F}(N, N-k|q)$. The arguments leading to such a conclusion can be found already in the classical paper by Andre Weil, Ref. [11], written in 1949. Incidentally, in his paper he studies the number of solutions in the field $\mathbf{F}_q$ for the Fermat hypersurface $\mathcal{F}$

$$a_0 z_0^{\hat{N}} + \cdots + a_{n+1} z_{n+1}^{\hat{N}} = 0, \tag{1.10}$$

living in the complex projective space $\mathbf{CP}^{n+1}$. Such a hypersurface was discussed in Part I (e.g. see Eq. (3.6)) in connection with our calculations of the Veneziano (and Veneziano-like) amplitudes. For the sake of space, with exception of Section 8.3, in this work we avoid the number-theoretic aspects related to the Veneziano amplitudes and their partition functions. We shall explain shortly below the rationale behind such an exception.

In the meantime, going back to our discussion, we notice that due to relation $m = N - k$ it is more advantageous for us to use parameters m and k than N and k. With this in mind we obtain,

$$\mathcal{F}(k, m|q) = \begin{bmatrix} k+m \\ k \end{bmatrix}_q = \frac{(q^{k+m}-1)(q^{k+m-1-1}-1)\cdots(q^{m+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = \prod_{i=1}^{k} \frac{1-q^{m+i}}{1-q^i}. \tag{1.11}$$

This result is reobtained in the main text using different mathematical arguments. It is of *central importance* for this work since it is obtainable from the supersymmetric partition function capable of reproducing the Veneziano and Veneziano-like amplitudes. In the limit: $q \to 1$ Eq. (1.11) reduces to the number $\mathcal{N}$ as required. To make connections with results already known in physics we need to rescale q's in Eq. (1.11), e.g. let $q = t^{1/i}$. Substitution of such an expression back into Eq. (1.11) and taking the limit $t \to 1$ again produces $\mathcal{N}$ in view of Eq. (1.5). This time, however, we can accomplish much more. By noticing that in Eq. (1.4) the actual value of N by design is not fixed thus far and taking into account that $m = N - k$ we can fix N by fixing m. Specifically, we would like to choose $m = 1 \cdot 2 \cdot 3 \cdots k$ and with such an m to consider a particular term in the product Eq. (1.11), e.g.

$$S(i) = \frac{1 - t^{1+(m/i)}}{1-t}. \tag{1.12}$$

In view of our "gauge fixing" the ratio m/i is a positive integer by design. This means that we are having a sum for the geometric progression. Indeed, if we rescale t again: $t \to t^2$, then we obtain

$$S(i) = 1 + t^2 + \cdots + t^{2\hat{m}}, \tag{1.13}$$

with $\hat{m} = \frac{m}{i}$. Written in such a form the sum above is just the Poincaré polynomial for the complex projective space $\mathbf{CP}^{\hat{m}}$. This can be seen by comparing pages 177 and 269 of the book by Bott and Tu, Ref. [9]. Hence, at least for some m's, *the Poincare′ polynomial for the Grassmannian in just the product of the Poincare′ polynomials for the complex projective spaces of known dimensionalities*. For m just chosen in the limit: $t \to 1$, we reobtain back the number $\mathcal{N}$ as required. This physically motivating process of gauge fixing we have just described can be replaced by more rigorous mathematical arguments. The recursion relation, Eq. (1.8), introduced earlier indicates that this is possible. The mathematical details leading to just described factorisation can be found, for instance, in the lecture notes by Schwartz, Ref. [12, Chapter 3]. Nevertheless, in Section 7, in view of their simplicity and intuitive appeal (as compared with arguments by Schwartz), we use different chain of arguments to arrive at the same conclusions. The topological significance of the Poincaré polynomial decomposition into the product of Poincaré polynomials is discussed in general terms in Section 4 and is used in Sections 7and 8 to recover the relevant physics. The relevant physics emerges by noticing that the partition function $Z(J)$ for the particle with spin J is given by [13]

$$Z(J) = \mathrm{tr}(\mathrm{e}^{-\beta H(\sigma)}) = \mathrm{e}^{cJ} + \mathrm{e}^{c(J-1)} + \cdots + \mathrm{e}^{-cJ} = \mathrm{e}^{cJ}(1 + \mathrm{e}^{-c} + \mathrm{e}^{-2c} + \cdots + \mathrm{e}^{-2cJ}), \tag{1.14}$$

where c is known constant. Evidently, up to a constant, $Z(J) \simeq S(i)$. But the result Eq. (1.14) is the Weyl character formula! This fact is to be discussed at length in Part III. The observation just made brings the classical group theory into our arguments. More importantly, because the partition function for the particle with spin J can be written in the language of $N = 2$ supersymmetric quantum mechanical model[8] as demonstrated by Stone [13] and others [14], the connections between the supersymmetry and the classical group theory are evident. We develop these connections further in this work. Part III (see also Ref. [6]) contains many additional results.

In view of arguments presented above, the Poincaré polynomial for the Grassmannian can be interpreted as a partition function for a kind of a spin chain made of spins of various magnitudes[9] caused by gauge fixing just described. In fact, the spin analogy is actually unnecessary since the formalism developed in Ref. [14] is valid for any finite dimensional homogenous space. It remains to demonstrate that the *finite dimensional* supersymmetric model just sketched can be used for reproduction of the Veneziano and Veneziano-like amplitudes. Such a task is accomplished in the rest of this work. The major reason for finite dimensionality is given by important theorems by Solomon, Shepard and Todd, Lefschetz and Serre discussed in the main text. In addition, the important theorem by Serre discussed in Section 9 not only provides needed support to our qualitative conclusions about finite dimensionality but also explains the connection between analytical properties of the Veneziano (and Veneziano-like) amplitude and the properties of space-time in which such amplitude "lives".

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[8] We hope that no confusion is made about the meaning of N in the present case.

[9] In such a context it can be vaguely considered as a variation on the theme of the Polyakov rigid string (Grassmann σ) model, Ref. [15], pp. 283–287, except that now it is *exactly solvable*. A somewhat different interpretation of the rigid string model was developed in our earlier work, Ref. [16].

1.2. Organization of the rest of this paper

The rest of this paper provides needed mathematical justifications supporting intuitive ideas just discussed. These justifications come mainly from the theory of invariants of finite pseudo-reflection groups. For the sake of uninterrupted reading, major ingredients of this theory are provided in the text along with important facts (in Appendix A) about the Weyl–Coxeter reflection groups and their generalization to pseudo-reflection groups by Shepard and Todd and others. Sections 2–6 contain all information needed for recovery of Eq. (1.11). In view of the recursion relation, Eq. (1.8), it can be interpreted as the Weyl character formula (this will be proven in Part III). In view of this observation, in Section 7 we accomplish several tasks. First, we provide needed mathematical justification for Eq. (1.12) thus connecting our results with those known earlier for spins and spin chains. Second, we investigate if this connection is the only option available or if there are other options. We find these other options as well. They allow us to bring into picture the formalism of exactly integrable systems and, in particular, to connect the obtained results with the tau function of the Kadomtsev–Petviashvili hierarchy. Through such a connection it is possible, in principle, to establish links with the existing string-theoretic formalism. Obtained results supply us with still other options however. This point of bifurcation from traditional formalism is studied further in Section 8. In it we discuss new exact solution of the Veneziano model. The obtained result happens to have additional uses which we also discuss in some detail. In particular, earlier searches for quantum mechanical systems whose spectrum reproduces zeros of the Riemann zeta function had resulted in the likely candidate: $H = xp$. In Part I, Eq. (1.12), does represents the Veneziano amplitude as product of zeta functions. In Section 8 we argue that this is not a curiosity: there is a deep reason for such a representation. Thus, we explain how the Hamiltonian $H = xp$ is related to the string-theoretic results we have obtained. After this, in Section 9 we discuss from various angles the important theorem by Serre. This theorem explains why there is not much freedom left to improve (replace, change) the Veneziano amplitude. This result is further strengthened by our observation that the function generating all Veneziano amplitudes is obtainable as a deformation retract for the Bergman kernel. Such a kernel has been used recently in connection with complex-hyperbolic geometry. The metric obtainable with such a kernel is an analog of the Lobachevskii metric in the real hyperbolic space. In accord with the ball model for the real hyperbolic space, there is analogous ball model for the complex hyperbolic space. The isometries of the boundary of such a ball model are described by the Heisenberg group. Since the real hyperbolic space is just a part of the complex-hyperbolic and since the real-hyperbolic is connected with the Minkowski space-time, we obtain the unusually tight connections between the Veneziano amplitudes and the properties of space-time in which these amplitudes act.

2. Selected exercises from Bourbaki (beginning)

In this section we begin our explanation of how group-theoretic methods can be used for reconstruction of both the Veneziano amplitudes and their generating/partition function. To accomplish this task, we need to work out some problems listed at the end of Chapter 5, paragraph 5 (problem set # 3) of the monograph by Bourbaki, Ref. [17]. Fortunately, answers to these problems to a large extent (but not completely!) can be extracted from the paper by Solomon [18]. In view of their crucial mathematical and physical importance, we reproduce some of his results in this section and will complete our treatment in Sections 3–6.

Let K be the field of characteristic zero (e.g. $\mathbf{C}$) and V be the vector space of finite dimension l over it. Let G be a subgroup of $GL_l(V)$ acting on V. Let q be the cardinality of G.[10] Introduce now the symmetric $S(V)$ and the exterior $E(V)$ algebra of V in order to construct invariants of the group G made of $S(V)$ and $E(V)$.[11] This task requires several steps. First, the multiplication of polynomials leads to the notion of a graded ring R.[12] For example, if the polynomial $P_i(x)$ of degree i belongs to the polynomial ring $\mathbf{F}[x]$, then the product $P_i(x)P_j(x) \in P_{i+j}(x) \in \mathbf{F}[x]$.

Definition 2.1. A graded ring R is a ring with decomposition $R = \oplus_{j=\mathbf{Z}} R_j$ compatible with addition and multiplication.

Next, for the vector space V if $x = x_1 \otimes \cdots \otimes x_s \in V^{\otimes_s}$ and $y = y_1 \otimes \cdots \otimes y_t \in V^{\oplus_t}$, then the product $x \otimes y \in V^{\oplus_{s+t}}$. A multitude of such type of tensor products forms the noncommutative associative algebra $T(V)$. Finally, the *symmetric* algebra $S(V)$ is defined by $S(V) = T(V)/I$, where the ideal I is made of $x \otimes y - y \otimes x$ (with both x and $y \in V$). In practical terms $S(V)$ is made of symmetric polynomials $\mathbf{F}[t_1, \ldots, t_l]$ with $t_1, \ldots, t_l$ being in one-to-one correspondence with the basis elements of V (that is each of t_i's is entering into $S(V)$ with power one). The exterior algebra $E(V)$ can be now defined analogously. For this we need to map the vector space V into the Grassmann algebra of V. In particular, if $x \in V$, then its image in the Grassmann algebra $\tilde{x}$ possess a familiar property: $\tilde{x}^2 = 0$. The graded two-sided ideal I can be defined now as

$$I = \{\tilde{x}^2 = 0 | x \to \tilde{x}; x \in V\}, \tag{2.1}$$

so that $E(V) = T(V)/I$. To complicate matters a little bit, we would like to consider a map d: $x \to \mathrm{d}x$ for $x \in V$ and $\mathrm{d}x$ belonging to the Grassmann algebra. If $t_1, \ldots, t_l$ is the basis of V, then $\mathrm{d}t_{i_1} \wedge \cdots \wedge \mathrm{d}t_{i_k}$ is the basis of $E_k(V)$ with $0 \leq k \leq l$ and, accordingly, the graded algebra $E(V)$ admits the following decomposition: $E(V) = \oplus_{k=0}^{l} E_k(V)$. Next, we need to construct the invariants of a (pseudo-reflection) group G made out of $S(V)$ and $E(V)$ and, more importantly, out of the tensor product $S(V) \otimes E(V)$. Toward this goal we need to determine if the action of the map $d : V \to E(V)$ extends to a differential map

$$d : S(V) \otimes E(V) \to S(V) \otimes E(V). \tag{2.2}$$

Clearly, $\forall x \in E(V)$ we have $d(x) = 0$. Therefore, $\forall x, y \in S(V) \otimes E(V)$ we can write $\mathrm{d}(x, y) = \mathrm{d}(x)y + x\,\mathrm{d}(y)$. By combining these two results together we obtain,

$$d : S_i(V) \otimes E_j(V) \to S_{i-1}(V) \otimes E_{j+1}(V), \tag{2.3}$$

i.e. the differentiation is compatible with grading. Now we are ready to formulate the theorem by Solomon [18] which is of central importance for our work. It is formulated in the form stated in Bourbaki, Ref. [17].

Theorem 2.2. Solomon [18] *Let $P_1, \ldots, P_k$ be algebraically independent polynomial forms made of symmetric combinations of $t_1, \ldots, t_k$ generating the ring $S(V)^G$ of invariants of G. Then,*

[10] i.e. the number of elements of G.

[11] The fact that we actually need to use the pseudo-reflection groups (e.g. see Appendix A, part A.4), will be explained mathematically only later, in Section 9. Hence, the formalism we are describing in this and following sections has actually a wider use.

[12] Surely, once the definition of such ring is given, there is no need to use polynomials. But in the present case this analogy is useful.

every invariant differential p-form $\omega^{(p)}$ may be written uniquely as a sum

$$\omega^{(p)} = \sum_{i_1<\cdots<i_p} c_{i_1\ldots i_p}\,\mathrm{d}P_{i_1}\cdots\mathrm{d}P_{i_p};\quad 1\le p\le k, \tag{2.4}$$

with $c_{i_1\ldots i_p}\in S(V)^G$. Moreover, actually, the differential forms $\Omega^{(p)}=\mathrm{d}P_{i_1}\wedge\cdots\wedge\mathrm{d}P_{i_p}$ with $1\le p\le k$ generate the entire algebra of G-invariants of $S(V)\otimes E(V)$.

Corollary 2.3. *Let $t_1,\ldots,t_k$ be the basis of V. Furthermore, let $S(V)=\mathbf{F}[t_1,\ldots,t_k]$ be its algebra of symmetric polynomials and $S(V)^G=\mathbf{F}[P_1,\ldots,P_l]$ its finite algebra of G-invariants.*[13] *Then, since $\mathrm{d}P_i=\sum_j\frac{\partial P_i}{\partial t_j}\mathrm{d}t_j$, we have*

$$\mathrm{d}P_1\wedge\cdots\wedge\mathrm{d}P_k = J(\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_k), \tag{2.5}$$

where, up to a constant factor $c\in K$, the Jacobian J is given by $J=c\Omega$ with

$$\Omega=\prod_{i=1}^{\nu}L_i^{c_i-1}. \tag{2.6}$$

In this equation L_i is the linear form defining i-th reflecting hyperplane H_i (it is assumed that the set of $H_1,\ldots,H_\nu$ of reflecting hyperplanes is associated with G), i.e. $H_i=\{\alpha\in V|L_i(\alpha)=0\}$ as defined in the Appendix A. In the same Appendix A, parts A.3, A.4, one finds that the set of all elements of G fixing H_i pointwise forms a cyclic subgroup of order c_i generated by pseudoreflections.

The result, Eq. (2.6), as well as the proportionality, $J=c\Omega$, can be found in the paper by Stanley [20]. It can be also found in much earlier paper by Solomon [18] where it is attributed to Steinberg and Shephard and Todd. Stanley's paper contains some details missing in earlier papers however.

Remark 2.4. The results given by Eqs. (2.5) and (2.6) play a key role in the theory of hyperplane arrangements to be briefly discussed in Section 9.

Using Theorem 2.2 by Solomon, Ginzburg proved the following

Theorem 2.5. Ginzburg, Ref. [21, p. 358] *Let $\omega_x(\xi_1,\xi_2)$ be a symplectic (Kirillov-Kostant) two-form (to be defined in Part III), let $\Omega^N=\omega_x^N$ be its N-th exterior power—the volume form, with N being the number of positive roots of the associated Weyl–Coxeter reflection group, then*

$$*(\Omega^N)=\mathrm{const}\cdot\mathrm{d}P_1\wedge\cdots\wedge\mathrm{d}P_k,$$

where the star $*$ denotes the standard Hodge-type star operator.

Corollary 2.6. *As it is argued in Part III, every nonsingular algebraic variety in projective space is symplectic. The symplectic structure gives raise to the complex Kähler structure which, in turn, is of the Hodge-type for the Kirillov-type symplectic manifolds. Alternative arguments leading to the same conclusion are presented in* Section 9.2.2.

[13] More details about $S(V)^G$ are given later in Section 9. The fact that the number of polynomial forms P_i is equal to the rank l of G is not a trivial fact. The proof can be found in Ref. [19, p.128]. Incidentally, this proof implies immediately the result, Eq. (2.6), given below. It is essential that to arrive at this result requires for the algebra of G-invariants to be finite.

In the famous paper, Ref. [22, p. 10, Eq.(4.1)]Atiyah and Bott argued that $\omega^{(p)}$ can be used for construction of the basis of the equivariant cohomology ring. Their results will be discussed in some detail in Part III. We refer our readers to the monograph [23] by Guillemin and Sternberg where the concepts of equivariant cohomology are pedagogically explained along with many other helpful mathematical facts of immediate relevance to our work.

The results just presented are essential for reconstruction of both the multiparticle Veneziano and Veneziano-like amplitudes. They also provide the needed mathematical background for adequate physical interpretation of these amplitudes. The next section illustrates these claims.

3. Veneziano amplitudes and Solomon's theorem

Let V be the complex affine space of dimension l and let $L_i(v)$, $v \in V$ be the linear form defining the ith hyperplane H_i, i.e.

$$H_i = \{v \in V | L_i(v) = 0\}, \qquad i = 1, \dots, l. \tag{3.1}$$

Results of Refs. [24,25] and those in Appendix A allow us to connect the set of hyperplanes, Eq. (3.1), with the complete fan (see also Section 9) and, using this fan, to associate it with it the polyhedron $\mathcal{P}$. Taking these facts into account, let us consider now an integral I of the type

$$I = \int_{\mathcal{P}} c_{i_1 \dots i_p} \mathrm{d}P_{i_1} \wedge \cdots \wedge \mathrm{d}P_{i_p}; \quad 1 \le p \le l. \tag{3.2}$$

Such an integral is connected with the period integrals of the type

$$\Pi(\lambda) = \oint_{\Gamma} \frac{P(z_1, \dots, z_n)}{Q(z_1, \dots, z_n)} \mathrm{d}z_1 \wedge \mathrm{d}z_2 \cdots \wedge \mathrm{d}z_n, \tag{3.3}$$

discussed in Part I. To avoid duplications, we are only presenting results of immediate relevance. In particular, in Part I we demonstrated that for the Fermat variety whose affine form is written as

$$\mathcal{F}_{\mathrm{aff}}(N) : Y_1^N + \cdots + Y_{n+1}^N = 1, \qquad Y_i = \frac{x_i}{x_0} \equiv z_i, \tag{3.4}$$

the period integral $\Pi(\lambda)$ is reduced (after calculation of the Leray residue) to the Veneziano (or Dirichlet)-like integral I[14] given explicitly by

$$I \doteq \int_{\Delta} t_1^{\langle c_1 \rangle / N - 1} \cdots t_{n+1}^{\langle c_{n+1} \rangle / N - 1} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n. \tag{3.5}$$

In view of Eqs. (2.5) and (2.6), it is of the type given by Eq. (3.2). In the present case the polyhedron $\mathcal{P}$ is the $n + 1$ simplex Δ. Not surprisingly, it is the deformation retract for the Fermat variety (as it is for $\mathbf{CP}^n$) since the Fermat variety is embedable into $\mathbf{CP}^n$ [24].[15] In accord with Part I, the symbol $\doteq$ denotes the statement: "with accuracy up to some constant (a phase factor)". The phase factors are important. We have discussed them at length in Part I without much theory behind them. Such a theory is well described, for example, in the monograph by Fulton, Ref. [24], and

[14] e.g. See Eq. (3.24) of Part I.

[15] We shall discuss this issue also in Section 9.

will be used and further discussed in Part III. In Section 9 we shall use some facts from this theory in order to prove that developments in this work do require use of the complex pseudo-reflection groups.

In connection with Part III (see also Ref. [6]) and in view of Theorems 2.2 and 2.5 we would like now to rederive result, Eq. (3.5), making emphasis on symplectic aspects of the Veneziano amplitudes. To this purpose, we would like to consider an auxiliary problem of calculation of the volume of k-dimensional simplex Δ_k. It is given by the integral of the type

$$\mathrm{vol}(\Delta_k) = \int_{x_i \geq 0} \mathrm{d}x_1 \cdots \mathrm{d}x_{k+1} \delta(1 - x_1 - \cdots - x_{k+1}). \tag{3.6}$$

Using results from symplectic geometry [26], it is straightforward to show that the above integral (up to unimportant constant) is just the microcanonical partition function for the system of $k+1$ harmonic oscillators whose total energy is equal to 1. To calculate such a partition function it is sufficient to take into account the integral representation of the delta function. Then, the standard manipulations with integrals produce the following anticipated result:

$$\mathrm{vol}(\Delta_k) = \frac{1}{2\pi} \oint \frac{\mathrm{d}y \, \exp(iy)}{(iy)^{k+1}} = \frac{1}{k!}. \tag{3.7a}$$

Clearly, for the dilated volume we would obtain instead: $\mathrm{vol}(n\Delta_k) = \frac{n^k}{k!}$, where n is the dilatation coefficient. This result was discussed already in Part I, e.g. see Eq. (1.24). This calculation allows us to obtain as well the volume of k-dimensional hypercube (or, perhaps more generally, the convex polytope $\mathcal{P}$) as

$$n^k = k!\mathrm{vol}(n\Delta_k). \tag{3.7b}$$

This result was obtained in famous paper by Atiyah [27] inspired by earlier result by Koushnirenko [28]. It is discussed at length both in Ref. [6] and Part III, in connection with alternative symplectic formulation of the partition function reproducing the Veneziano (and Veneziano-like) amplitudes. In the meantime, the observations just made allow us to rederive the Veneziano amplitude in a much simpler way. To do so, we extend our analysis of Eq. (3.6) having in mind both Theorems 2.2 and 2.5. This leads us to consideration of the integral of the type

$$I = \int_{x_i \geq 0} \mathrm{d}x_1^{\langle c_1 \rangle} \wedge \cdots \wedge \mathrm{d}x_{k+1}^{\langle c_{k+1} \rangle} \delta(1 - x_1^N - \cdots - x_{k+1}^N), \tag{3.8}$$

written in accord with notations of Part I. The presence of δ function reminds us about the procedure (discussed in Part I) of taking the Leray-type residue in the period integral, Eq. (3.3). We would like to demonstrate now that the integral, Eq. (3.8) can be calculated much easier as compared to calculations described in detail in Part I. To simplify notations, let $\langle c_i \rangle = p_i$ and, furthermore, let $n_i = \frac{N}{p_i}$. By analogy with (3.7a) we obtain (up to a constant as before),

$$I \doteq \frac{1}{2\pi} \oint \mathrm{d}y \left[\prod_{i=1}^{k+1} \left(\frac{1}{iyn_i} \right)^{1/n_i} \Gamma\left(\frac{1}{n_i} \right) \right] \exp(iy) \doteq \frac{\Gamma\left(\frac{p_1}{N}\right) \cdots \Gamma\left(\frac{p_{k+1}}{N}\right)}{\Gamma\left(\sum_i \frac{p_i}{N}\right)}, \tag{3.9}$$

in agreement with Eq. (3.27) of Part I as required.

4. Selected excersises from Bourbaki (continuation)

The results of previous section indicate that the Veneziano and Veneziano-like amplitudes can be reconstructed from the algebra of invariants $(S(V) \otimes E(V))^G$ of the group G not yet specified. The question naturally arises: can we use the same invariance principle in order to reconstruct the meaningful physical model reproducing the Veneziano and Veneziano-like amplitudes? We provide positive answer to the above posed question in this section and in Sections 6–9.

To begin, we need to discuss properties of the ring $S(V)^G$ of symmetric invariants composed of algebraically independent polynomial forms $P_1, \ldots, P_l$ made of symmetric combinations of $t_1, \ldots, t_l$ raised to some powers d_i $(i = 1, \ldots, l)$ different for different reflection groups [29]. The ring of invariants is graded and it admits a decomposition (which actually is always finite): $S(V)^G = \bigoplus_{j=0}^{\infty} S_j(V)^G$. Provided that $\dim_K V_j^G$ is the dimension of the graded invariant subspace $S_j(V)^G$ defined over the field K, the following definition can be given.

Definition 4.1. The Poincaré polynomial $P(S(V)^G, t)$ is defined by

$$P(S(V)^G, t) = \sum_{i=0}^{\infty} (\dim_K V_j^G) t^i. \tag{4.1}$$

The Poincaré polynomial possesses the splitting property [29] (the most useful for applications in K-theory [12]). This can be described as follows. If the total vector space M is made as a product $V \otimes_K V'$ of vector spaces V and V', then the Poincaré polynomial of such a product is given by

$$P(V \otimes_K V', t) = P(V, t) P(V', t). \tag{4.2}$$

This splitting property is of topological nature [9] since it reflects the decomposition property of the total topological space into pieces and for this reason is extremely useful in actual calculations. In particular, let us consider the polynomial ring $F[x]$ made of monomials of degree d which are the building blocks of the graded vector space V as discussed in Section 2. The Poincaré polynomial for such a space is given by

$$P(V, t) = 1 + t^d + t^{2d} + \cdots = \frac{1}{1 - t^d}. \tag{4.3}$$

Consider now the multivariable polynomial ring $F[x_1, \ldots, x_n]$ made of monomials of respective degrees d_i. Then, using the splitting property, we obtain at once

$$V^T = F[x_1] \otimes_K F[x_2] \otimes_K F[x_3] \otimes_K \cdots \otimes_K F[x_n]$$

and, of course,

$$P(V^T, t) = \frac{1}{1 - t^{d_1}} \cdots \frac{1}{1 - t^{d_n}}. \tag{4.4}$$

In particular, if all d_i in Eq. (4.4) are equal to one, which is characteristic for $S(V)$ defined in Section 2, e.g. read Ref. [29, p. 171], then we reobtain back Eq. (1.22) of Part I.

Remark 4.2. Since the Laplace transform of Eq. (1.22) of Part I produces the (nonsymmerized) Veneziano amplitude, the connection between the theory of invariants of finite (pseudo)reflection groups and the physical model reproducing such an amplitude becomes apparent already at this point.

This remark allows us to make several additional steps to make our presentation mathematically self contained and focused on physics. To this purpose, let $G \subset GL(V)$ be one of such reflection groups. Suppose that its cardinality $|G| = \prod_i d_i$. Next, we introduce the averaging operator $Av : V \to V$ via

$$Av(x) = \frac{1}{|G|} \sum_{\varphi \in G} \varphi \circ x. \tag{4.5}$$

By definition, x is the group invariant, $x \in V^G$, if $Av(x) = x$. In particular,

$$\dim_K V_j^G = \frac{1}{|G|} \sum_{\varphi \in G} \mathrm{tr}(\varphi_j). \tag{4.6}$$

This result can be explained as follows. Suppose $x \in V^G$, then Av $(Av(x) = x) \to Av^2(x) = Av(x) = x$. Thus, the Av operator is indepotent. Such indepotent operator has evidently only two eigenvalues: 1 and 0. Using this fact in Eq. (4.5) produces Eq. (4.6).

By combining Eq. (4.6) result with Eq. (4.1) we obtain,

$$\begin{aligned} P(S(V)^G, t) &= \sum_{i=0}^{\infty} (\dim_K V_j^G) t^i = \sum_{i=0}^{\infty} \frac{1}{|G|} \sum_{\varphi \in G} \mathrm{tr}(\varphi_i) t^i \\ &= \frac{1}{|G|} \sum_{\varphi \in G} \left[\sum_{i=0}^{\infty} \mathrm{tr}(\varphi_i) t^i \right] = \frac{1}{|G|} \sum_{\varphi \in G} \frac{1}{\det(1 - \varphi t)}. \end{aligned} \tag{4.7}$$

The obtained result is known as the Molien theorem [29]. It is based on the following nontrivial identity

$$\sum_{i=0}^{\infty} \mathrm{tr}(\varphi_i) t^i = \frac{1}{\det(1 - \varphi t)}, \tag{4.8}$$

valid for the upper triangular matrices, i.e. for matrices which belong to the Borel subgroup B of G.[16] For such matrices

$$\mathrm{tr}(\varphi_i) = \sum_{j_1 + j_2 + \cdots + j_n = i} \lambda_1^{j_1} \cdots \lambda_n^{j_n}, \tag{4.9}$$

where the Borel-type matrix φ of dimension n has $\lambda_1, \ldots, \lambda_n$ on its diagonal. Substitution of Eq. (4.9) into Eq. (4.7) produces

$$\begin{aligned} \sum_{i=0}^{\infty} \mathrm{tr}(\varphi_i) t^i &= \sum_{i=0}^{\infty} \left[\sum_{j_1 + j_2 + \cdots + j_n = i} \lambda_1^{j_{10}} \cdots \lambda_n^{j_n} \right] t^i = \left[\sum_{j_1=0}^{\infty} \lambda_1^{j_1} t^{j_1} \right] \cdots \left[\sum_{j_n=0}^{\infty} \lambda_n^{j_n} t^{j_n} \right] \\ &= \frac{1}{1 - \lambda_1 t} \cdots \frac{1}{1 - \lambda_n t} = \frac{1}{\det(1 - \varphi t)}. \end{aligned} \tag{4.10}$$

We have gone through all details in order to demonstrate the *bosonic nature* of the obtained result: by replacing t with $\exp(-\varepsilon)$ with $0 \leq \varepsilon \leq \infty$ and associating numbers j_i with the Bose

[16] In Part III we shall provide important details on the subgroup B.

statistic occupation numbers we have obtained the partition function for the set of n independent harmonic oscillators (up to zero point energy). This result will be reobtained in Part III using different arguments.

In view of Eq. (4.7), thus obtained result should be additionally group averaged. In particular, by combining Eqs. (4.4) and (4.7) we obtain,

$$P(S(V)^G, t) = \frac{1}{|G|}\sum_{\varphi\in G}\frac{1}{\det(1-\varphi t)} = \prod_{i=1}^{n}\frac{1}{1-t^{d_i}}. \tag{4.11}$$

This result is valid for any (pseudo)reflection group $G \subset GL(V)$ whose cardinality $|G| = \prod_i d_i$ in accord with the conditions of Theorem 2.2 by Solomon. Following Humphreys [30], it is useful to reinterpret Eq. (4.11) using the following physically motivated arguments. We consider an action of the averaging operator, Eq. (4.5), on the monomials

$$x = z_1^{j_1}\cdots z_n^{j_n}, \quad \text{where} \quad j_1 + j_2 + \cdots + j_n = i.$$

These are the eigenvectors for φ_i with the corresponding eigenvalues $\lambda_1^{j_1}\cdots\lambda_n^{j_n}$. The weighted sum of these eigenvalues is the trace of the linear operator $Av(x)$ acting on monomials from $S_i(V)^G$. But, according to Eq. (4.6), this is just the dimension of space $S_i(V)^G$. This dimension has the following physical meaning. If for the moment we assume (and later, in Section 9, we prove) that all eigenvalues in Eq. (4.9) are i-th roots of unity, then, by combining Eqs. (9.15)–(9.18) of Section 9 and the results of Appendix A, parts A.3, A.4 we arrive at the Veneziano condition

$$\sum_k m_k j_k = i, \tag{4.12a}$$

again. Since in this equation $m_i = d_i - 1 \bmod i$, it is equivalent to

$$\sum_k d_k j_k = 0 \bmod i. \tag{4.12b}$$

In particular, if $d_k j_k = \omega i$ (for $k = 1, \dots, n$ and $\omega \in \mathbf{Z}$) then, using this equation along with Eqs. (4.8) and (4.9), we arrive at the following result:

$$\begin{aligned}\sum_{i=0}^{\infty}\operatorname{tr}(\varphi_i)t^i &= \sum_{i=0}^{\infty}\left[\sum_{j_1d_1+j_2d_2+\cdots+j_nd_n=i}\lambda_1^{j_1d_1}\cdots\lambda_n^{j_nd_n}\right]t^i \\ &= \sum_{j_1=0}^{\infty}t^{j_1d_1}\sum_{j_2=0}^{\infty}t^{j_2d_2}\cdots\sum_{j_n=0}^{\infty}t^{j_nd_n} = \prod_{i=1}^{n}\frac{1}{1-t^{d_i}},\end{aligned} \tag{4.13}$$

to be compared with earlier obtained Eq. (4.11). Again, we have gone through all these details in order to demonstrate *the bosonic nature* of the obtained result, Eq. (4.11). Clearly, the result, Eq. (4.11), can be reproduced using the path integrals for n independent bose-like particles, e.g harmonic oscillators. Such a conclusion is going to be strengthened in Part III devoted to the symplectic interpretation of the obtained results. However, the obtained results are incomplete since thus far we were dealing only with $S(V)^G$-type of invariants made of monomials raised to d_i-th powers. Theorem 2.2 requires us to construct the Poincaré polynomials for invariants of $(S(V)\otimes E(V))^G$type. To design such polynomials we need to discuss several additional topics. These are presented in the next two sections.

5. Additional facts from the theory of pseudo-reflection groups

In Appendix A, part A.4, we have listed some basic facts about pseudo-reflection groups. At this point we would like to extend this information. To this purpose we would like to use some results from the classical paper by Shepard and Todd [31] (S-T).We shall use these results along with those from the monograph by McMullen [32] containing the up to date developments related to the S-T work.

Adopting S-T notations, let $N \geq 1$, $n \geq 2$, and let p be a divisor of N, i.e. $N = pq$. In addition, let ξ be a primitive N-th root of unity. Then, the unitary group $G(N, p, n)$ is defined as the group of all monomial transformations in $\mathbf{C}^n$ of the form (e.g. see Section 9 below)

$$x_i' = \xi^{\nu_i} x_{\sigma(i)}, \qquad i = 1, \ldots, n, \tag{5.1}$$

where $\sigma(1), \ldots, \sigma(n)$ is permutation σ of $(1, \ldots, n)$, i.e. $\sigma \in S_n$, and

$$\sum_i \nu_i = 0(\mathrm{mod} N). \tag{5.2a}$$

In the case if $N = pq$ the above condition should be changed to

$$\sum_i \nu_i = 0(\mathrm{mod} p). \tag{5.2b}$$

The group $G(N, p, n)$ has order (or cardinality) $|G| = qN^n n!$. The order of the group $G(N, p, n)$ can be determined by considering the set of 2-fold reflections given by

$$x_i' = \xi^{\nu} x_j, \qquad x_j' = \xi^{-\nu} x_i, \qquad x_k' = x_k, \qquad k \neq i, j. \tag{5.3}$$

Such set generates the normal subgroup of order $N^n n!$. The other reflections, if any, are of the form

$$x_j' = \xi^{\nu N/r} x_j, \qquad x_i' = x_i \quad \text{for } i \neq j, \tag{5.4}$$

where $j = 1, \ldots, n$, $(\nu, N/r) = 1$ and $r|q$ if $q > 1$. The following theorem can be found in McMullen's book [32, p. 292].

Theorem 5.1. *If $n \geq 2$, then up to conjugacy within the group of all unitary transformations, the only finite irreducible unitary reflection groups in $\mathbf{C}^n$ which are imprimitive are the groups $G(m, p, n)$ with $m \geq 2$, $p|m$ and $(m, p, n) \neq (2, 2, 2)$.*

Definition 5.2. A group G of unitary transformations of $\mathbf{C}^n$ is called *imprimitive* if $\mathbf{C}^n$ is the direct sum $\mathbf{C}^n = E_1 \oplus \cdots \oplus E_k$ of non-trivial proper linear subspaces $E_1, \ldots, E_k$ such that the family $\{E_1, \ldots, E_k\}$ is invariant under G.

For the purposes of this work, it is sufficient to consider only the case $p = 1$. The group $G(N, 1, n)$, traditionally denoted as γ_n^N, is the group of symmetries of the complex n-cube. Actually, it is the same as that for the real n-cube [29,31] which is inflated by the factor of N. In the standard notations [30] the cubic symmetry is denoted as B_{n+1} while in the S-T notations it corresponds to just mentioned group $G(N, 1, n)$ whose exponents $d_i = Ni$ with $i = 1, \ldots n$ [29,31]. In the typical case of *real* space with cubic symmetry we have $N = 2$ (e.g. see Eq. (A.6) of Appendix A) so that these exponents d_i coincide with those for B_{n+1} in accord with the exponents for this group listed in the book by Humphreys [30, p. 59], as required.

In order to utilize all these observations efficiently, we need to reobtain the same results from another perspective. To this purpose, following S-T, we introduce an auxiliary function

$g_r(m, p, n)$ describing the number of reflexion operations made with help of $G(m, p, n)$ which leave fixed every point of the subspace of dimensionality $n - r$, $r = 0, 1, \ldots, n$.[17] By definition, $g_0(m, p, n) = 1$. Moreover, let

$$G(m, p, n; t) \equiv \sum_{r=0}^{n} g_r(m, p, n) t^r. \tag{5.5}$$

On one hand, the above equation serves as the definition of the generating function $G(m, p, n; t)$, on another, in view of Eq. (1.8) of Section 1, we can reinterpret the r.h.s. as the Weyl character formula. The full proof of this fact is given in Part III. Shepard and Todd calculate $G(m, p, n; t)$ explicitly. Their derivation is less physically adaptable however than that obtained by Solomon, Ref. [18]. Hence, we would like to discuss Solomon's results now.

6. Selected exercises from Burbaki (end)

Our main objective at this point is to obtain the explicit form of $G(m, p, n; t)$ defined in Eq. (5.5) and to explain its physical meaning. To this purpose, let us recall that according to Theorem 2.2 the differential form $\omega^{(p)}$, Eq. (2.4), belongs to the set of G-invariants of the product $S(V) \otimes E(V)$. The splitting property, Eq. (4.2), of the Poincaré polynomials requires some minor changes for the present case. In particular, if by analogy with $S(V)^G$ decomposition we would write $(S(V) \otimes E(V))^G = \bigoplus_{i,j} S_i(V)^G \otimes E_j(V)^G$, then the associated Poincaré polynomial can be defined by

$$P((S(V) \otimes E(V))^G; x, y) = \sum_{i,j \geq 0} (\dim_K S_i^G \otimes E_j^G) x^i y^j. \tag{6.1}$$

Following Solomon [18], by analogy with Eq. (4.6) we introduce

$$\dim_K (S_i^G \otimes E_j^G) = \frac{1}{|G|} \sum_{\varphi \in G} \mathrm{tr}(\varphi_i) \mathrm{tr}(\varphi_j). \tag{6.2}$$

In order to use this result we need to take into account that

$$\sum_{j=0}^{n} \mathrm{tr}(\varphi_j) y^j = \det(1 + \varphi y), \tag{6.3}$$

to be contrasted with Eq. (4.8). To prove that this is indeed the case it is sufficient to recall that for fermions the occupation numbers j_i are just 0 and 1. Hence, in view of Eq. (4.13), but accounting for the fermionic nature of the occupation numbers in the present case, we obtain,

$$\sum_{j=0}^{n} \mathrm{tr}(\varphi_j) y^j = \sum_{j=0}^{n} \left[\sum_{j_1 + j_2 + \cdots + j_n = j} \lambda_1^{j_1} \cdots \lambda_n^{j_n} \right] y^j = \prod_{i=1}^{n} \sum_{j_i=0}^{1} \lambda_i^{j_i} y^{j_i} = \prod_{i=1}^{n} (1 + \lambda_i y). \tag{6.4}$$

As in the bosonic case, the result, Eq. (6.4), can be obtained using the fermionic path integrals for n independent particles (say, fermionic oscillators) obeying the Fermi-type statistics. Using Eq.

[17] This construction reminds us about the Grassmannians considered in Section 1. We shall take the full advantage of this observation momentarily.

(6.2) in (6.1) and taking into account the rest of the results obtained in Section 4, the following expression for the Poincaré polynomial is obtained

$$P((S(V)\otimes E(V))^G;x,y)=\frac{1}{|G|}\sum_{\varphi\in G}\frac{\det(1+\varphi y)}{\det(1-\varphi x)}=\prod_{i=1}^{n}\frac{1+yx^{d_i-1}}{1-x^{d_i}},\tag{6.5}$$

in accord with Bourbaki [17]. To check its correctness we can: (a) put $y=0$ thus obtaining back Eqs. (4.4) and (4.11) or, (b) put $y=-x$ thus obtaining identity $1=1$ between the second and the third terms above.

Remark 6.1. Since the result, Eq. (6.5), is just the ratio of determinants, its supersymmetric nature should be clear to everybody familiar with path integrals.

Eq. (6.5), can now be used for several tasks. First, for completeness of presentation, we would like to recover the major S-T result:

$$G(m,p,n;t)=\prod_{i=1}^{n}(m_it+1),\tag{6.6}$$

extensively used in theory of hyperplane arrangements [33,34] to be discussed further in Section 9. Taking into account notations introduced in Eqs. (4.12), the above equation produces (for $t=1$) the following result: $G(m,p,n;t=1)=|G|=\prod_{i=1}^{n}d_i$, which is in accord with Section 4, as required. Moreover, in view of Eq. (5.5), it allows us to recover $g_r(m,p,n)$. After this is done, we need to discuss its physical meaning.

To recover the S-T results let us rewrite Eq. (6.5) as follows

$$\sum_{\varphi\in G}\frac{\det(1+\varphi y)}{\det(1-\varphi x)}=|G|\prod_{i=1}^{n}\frac{(1+yx^{d_i-1})}{(1-x^{d_i})}\tag{6.7}$$

and let us treat the right (R) and the left (L) hand sides separately. Following Bourbaki [17], we put $y=-1+t(1-x)$. Substitution of this result to R produces at once

$$R|_{x=1}=\prod_{i=1}^{n}(d_i-1+t)=\prod_{i=1}^{n}(m_i+t).\tag{6.8}$$

To do the same for L requires us to keep in mind that $\det AB=\det A\det B$ and, hence, $\det AA^{-1}=1$ leads to $\det A^{-1}=1/\det A$. Therefore, after few steps we arrive at

$$L|_{x=1}=\sum_{r=1}^{n}h_rt^r.\tag{6.9}$$

Equating L with R, replacing t by $1/T$ and relabeling $1/T$ again by t and h_l by $\tilde{h}_l=g_r(m,p,n)$ we obtain the S-T result, Eq. (5.5). To obtain physically useful result we have to take into account that for the cubic symmetry we had quoted already in the previous section the result: $d_i=iN$. Therefore, let $y=-x^{Nq+1}$ in Eq. (6.5), then we obtain,

$$P((S(V)\otimes E(V))^G;z)=\prod_{i=1}^{n}\frac{1-z^{q+i}}{1-z^i}.\tag{6.10}$$

Remark 6.2. The result almost identical to our Eq. (6.10) was obtained some time ago in the paper by Lerche et al. [35, p. 444, Eq. (4.4)]. To obtain their result, it is sufficient to replace

z^{q+i} by z^{q-i}. Clearly, such a substitution is *not permissible* in our case. Nevertheless, some ideas discussed in the paper by Lerche et al. happen to be helpful in obtaining the correct partition function for the Veneziano (and Veneziano-like) amplitudes. This will be discussed in detail in the following two sections.

Taking into account the cubic symmetry in Eq. (6.10) in the limit $z = x^N = 1$ we obtain,

$$P((S(V) \otimes E(V))^G; z = 1) = \frac{(q+1)(q+2)\cdots(q+n)}{n!}. \tag{6.11}$$

Both Eqs. (6.10) and (6.11) have been obtained in Section 1 in a much simpler way so that there is no need to repeat the arguments presented there.

From Section 1 we know already that $P((S(V) \otimes E(V))^G; z)$ given by Eq. (6.10) is the Poincaré polynomial for the complex Grassmann manifold. The question arises: is the method of obtaining such a polynomial specific only for Grassmannians? The answer is clearly "No"! This had been demonstrated mathematically rigorously by Hiller, Ref. [36], based on earlier fundamental results by Bernstein, Gelfand and Gelfand, Ref. [37] (BGG). Incidentally, Hiller *does* obtain our main result, Eq. (6.10), using BGG formalism, e.g. see Ref. [36, p. 155 (top)]. His derivation is entirely different and is considerably more complicated than ours.[18] The invariant algebra $(S(V) \otimes E(V))^G$ considered by Solomon, Ref. [18], is called "the topological algebra" in Ref. [38]. This reference explains in detail the universal nature of such an algebra which makes it absolutely indispensable in the theory of fiber bundles, *K*-theory, theory of characteristic classes and equivariant cohomology. We discuss this topic further in Section 9.2.2.

7. Designing Veneziano partition function using algebraic geometry

7.1. General considerations

The paper by Lerche et al., Ref. [35], provides plausible arguments implying that (up to some unimportant constant) *any* one-variable Poincaré polynomial can be actually interpreted as some kind of the Weil character formula. In Part III we shall reach the same conclusions using entirely different arguments. These are presented in conjunction with the symplectic development of our formalism. To actually use our result, Eq. (6.10), (or, which is the same, Eq. (1.11)) we do need to take into account the connection with the Weyl character formula just mentioned. The recursion relation, Eq. (1.8), provides already an indication that, indeed, the product given in the r.h.s. of Eq. (1.11) is in fact a polynomial in q whose highest degree is km. Such a polynomial can be already interpreted as the Weyl character formula. In principle, because of the noticed connections with the Weyl character formula, one can use the supersymmeric formalism for reproduction of this formula. Such formalism, developed for any homogenous space (including that for the Grassmanniann) in Refs. [13,14], can be used for reproduction of the result Eq. (6.10). In view of the results to be presented in the next section, this is not the most illuminating way however to arrive at our final destination. Such a derivation will not take into account the "gauge" freedom and the "gauge fixing" discussed in Section 1. In view of this discussion, it is more advantageous to take advantage of the fact that the Grassmannian can be mapped into the product of complex projective spaces of prescribed dimensionalities. At the level of classifying spaces such a possibility is indicated on page 303 in the book by Bott and Tu, Ref. [9], and proven

[18] We encourage our readers to make such a comparison.

in the book by Husemoller, Ref. [39, p. 297, Proposition 3.1]. There is however another, more direct, way to obtain the desired result without recourse to the classifying space. In Section 1 we indicated that such a possibility does indeed exist: to this purpose it is sufficient to "fix the gauge" by choosing $m = 1 \cdot 2 \cdots k$ in Eq. (1.11). Then, the Poincaré polynomial for the complex Grassmannian becomes manifestly decomposable into the product of Poincaré polynomials for the complex projective spaces of prescribed dimensionalities. Although such a decomposition is plausible it is a bit restricted. Below we would like to discuss another less restricted way to arrive at the desired result. By doing so we shall accomplish several tasks. First, we can then formally use the results by Stone, Ref. [13], for the partition function for a particle with spin in the magnetic field. Second, and more important for us, by embedding the complex Grassmannian into the complex projective space of prescribed dimensionality we shall obtain important results to be used in the rest of this paper.

To embed the Grassmannian into the complex projective space requires several steps. We would like to describe them now.

7.2. The Plücker embedding

For reader's convenience we would like to summarize the idea behind such an embedding. In particular, let V be an n dimensional vector space and let $E(V)$ be its exterior algebra, as described in Section 2, so that $E(V) = \oplus_{k=1}^{l} E_k(V)$ where $1 \leq k \leq l$. While the space $E_l(V)$ is one dimensional, the dimensionality of the subspace $E_k(V)$, is known to be $\mathcal{N} = \dim E_k(V) = \begin{pmatrix} l \\ k \end{pmatrix}$. This should be contrasted with the dimensionality of the usual vector subspace which is just k. The number just produced reminds us about the number of subspaces of the (complex) Grassmannian $G(k, l)$ we have discussed in Section 1. This means that the totality of k-dimensional subspaces of l-dimensional vector space V can be identified with $\dim E_k(V)$. Let now $\hat{E}_k(V)$ be a particular member of the exterior algebra so that if $\mathbf{w}_1, \ldots, \mathbf{w}_k$ represents a set of linearly independent vectors defining the chosen k-dimensional subspace, then $\hat{E}_k(V) = \mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$. Clearly, if $\mathbf{w}_i \in \mathbf{C}_i$, then the product $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$ represents some point in $\mathbf{C}^{\mathcal{N}}$ and, at the same time, it represents some particular subspace of the vector space V and, as such, can be used to describe the Grassmannian. Moreover, if $\mathbf{e}_1, \ldots, \mathbf{e}_l$ is an ordered basis for V, then there are some $k \times l$ matrices $\mathbf{M} = (a_{i,j})$ such that

$$\mathbf{w}_j = \sum_{i=1}^{l} a_{i,j} \mathbf{e}_i, \tag{7.1}$$

and, accordingly,

$$\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k = \sum_{(i_1,\ldots,i_k)} m_{i_1,\ldots,i_k} \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}, \tag{7.2}$$

where $m_{i_1,\ldots,i_k}$ is made out of $k \times k$ minors of the matrix $\mathbf{M}$ formed by columns of $\mathbf{M}$ with indices $i_1, \ldots, i_k$. Since both $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$ and $\mathbf{e}_{i_1} \wedge \cdots . \wedge \mathbf{e}_{i_k}$ are some points in $\mathbf{C}^{\mathcal{N}}$, Eq. (7.2) can be interpreted as an equivalence relation and thus provides a projective embedding of the Grassmannian into $\mathbf{CP}^{\mathcal{N}-1}$. The r.h.s. of Eq. (7.2) represents a kind of a basis expansion of the tensor $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$ so that $m_{i_1,\ldots,i_k}$ are some (actually, Plücker) coordinates with respect to the standard basis. Suppose we have another tensor $\mathbf{w}'_1 \wedge \cdots \wedge \mathbf{w}'_k$ and interested in relating it with

$\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$. This can be achieved if there is another $k \times k$ matrix $\mathbf{A}$ such that $\mathbf{M} = \mathbf{A}\mathbf{M}'$ and, accordingly,

$$m_{i_1,\ldots,i_k} = [\det \mathbf{A}] m'_{i_1,\ldots,i_k}. \tag{7.3}$$

Finally, following Fulton, Ref. [40, p. 108], the Sylvester theorem (discovered in 1851) should be used to arrive at Plücker relations. These relations provide guarantee that such an embedding of the Grassmannian is permissible. To describe the Sylvester theorem we should notice that, actually, by symmetry the matrix $\mathbf{A}$ is of the same kind as the $k \times k$ minor of $\mathbf{M}$. That is it should be a $k \times k$ matrix. Because of this, the Sylvester theorem can be stated as follows

Theorem 7.1 (Sylvester). *Let $\mathbf{M}$ and $\mathbf{N}$ be any $k \times k$ matrices and let $1 \le \lambda \le k$, then*

$$[\det \mathbf{M}][\det \mathbf{N}] = \sum \det \mathbf{M}' \det \mathbf{N}', \tag{7.4}$$

where the sum is taken over all pairs of matrices $\mathbf{M}'$ and $\mathbf{N}'$ obtained from $\mathbf{M}$ and $\mathbf{N}$ by interchanging a fixed set of λ columns of $\mathbf{N}$ with any λ columns of $\mathbf{M}$, preserving the ordering of the columns.

This concludes our description of the Plücker embedding. For the goals we would like to accomplish, such an embedding is not sufficient. Hence, now we would like to discuss the Segre embedding.

7.3. The Segre embedding

The idea of this kind of embedding is rather simple and has its origins in a simple problem which can be formulated as follows. If the complex space $\mathbf{C}^2$ can be thought of as a Cartesian product $\mathbf{C}^1 \times \mathbf{C}^1$, is it possible to construct, say, $\mathbf{CP}^2$ as $\mathbf{CP}^1 \times \mathbf{CP}^1$? Our experience with the Plücker embedding suggests that this may be possible if we look at the operation "$\times$" group-theoretically. Specifically, let $\{z_0, \ldots, z_n\}$ represent a point in $\mathbf{CP}^n$ while $\{z'_0, \ldots, z'_m\}$ represent a point in $\mathbf{CP}^m$,[19] then the Segre embedding $s_{n,m} : \mathbf{CP}^n \times \mathbf{CP}^m \to \mathbf{CP}^N$, $N = (n+1)(m+1) - 1$, is described explicitly as

$$s_{n,m} : (\{z_0, \ldots, z_n\}, \{z'_0, \ldots, z'_m\}) \to (\{\cdots, z_i z'_j, \cdots\}). \tag{7.5}$$

Let $R_{ij} = z_i z'_j$, then the analogs of Plücker relations in the present case are the conditions

$$R_{ij} R_{kl} = R_{il} R_{kj}. \tag{7.6}$$

Finally, we need to describe the Veronese embedding.

7.4. The Veronese embedding

It can be described as follows. Let $\{z_0, \ldots, z_k\}$ be a point in $\mathbf{CP}^k$ and let v_n be the map $v_n : \mathbf{CP}^k \to \mathbf{CP}^{\mathcal{N}}$, where $\mathcal{N} = \begin{pmatrix} n+k \\ k \end{pmatrix} - 1$[20], explicitly described as

$$v_n : \{z_0, \ldots, z_k\} \to \{x_0, \ldots, x_{\mathcal{N}}\}, \tag{7.7}$$

---

[19] One should keep in mind that points in the projective space are the equivalence classes. The notations in the text refer to some set of coverings of $\mathbf{CP}^n$ by $\mathbf{C}^{n+1'}s$ such that at least one of z_i's is strictly nonzero, as usual.

[20] In the literature on algebraic geometry, e.g. see Ref. [41], one finds an alternative way of writing $\mathcal{N}$, e.g. $\mathcal{N} = \begin{pmatrix} n+k \\ n \end{pmatrix} - 1$ but, in view of Eq. (1.5), both are the same numbers.

with $x_i = x_0^{i_0} x_1^{i_1} \cdots x_k^{i_k} \equiv X^{\mathbf{I}}$ and $i_0 + \cdots + i_k = n$, then this is the Veronese map provided that for any quadruple of multiindices $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{L}$ the following relation

$$X^{\mathbf{I}} X^{\mathbf{J}} = X^{\mathbf{K}} X^{\mathbf{L}}, \tag{7.8}$$

holds.

7.5. From analysis to synthesis

Being armed with these descriptions of the respective embeddings we are ready now to use all three of them. We begin with observation that the dimensionalities of projective spaces in the case of Plücker and Veronese embeddings can be made the same due to the same type of combinatorics in both cases. This happens when we require: (Veronese) $n + k = l$ (Plücker). Evidently, relations of the type given by Eq. (7.8) will be also satisfied by the Plücker relations (since in both cases we are dealing with the same multiindex sets). Hence, we can identify point by point both projective spaces. It should be clear that this can be done only with some restriction on ordering of indices but this is sufficient for our physical purposes. We shall discuss this topic further in the next subsection. Next, we can think about making a projective space of dimensionality $\mathcal{N}$ out of projective spaces of smaller dimensionality using the Segre embedding. This fact is important physically since it is connected with the fusion rules for scattering (e.g. Veneziano) amplitudes. Suppose, we would like to compose a larger space out of complex projective spaces of dimensionalities $i_0, \ldots, i_k$. Then, the Segre embedding can be described schematically as the follows:

$$\begin{aligned} &V_{i_0} \times \cdots \times V_{i_k} \to V_{i_0} \otimes \cdots \otimes V_{i_k} \text{ causing,} \\ &\mathbf{CP}(V_{i_0}) \times \cdots \times \mathbf{CP}(V_{i_k}) \hookrightarrow \mathbf{CP}(V_{i_0} \otimes \cdots \otimes V_{i_k}), \end{aligned} \tag{7.9}$$

with dimensionality of the final complex projective space being equal to $\mathcal{N} = (i_0 + 1)(i_1 + 1) \cdots (i_k + 1) - 1$. To compare this dimensionality with that for, say, the Veronese-type space we have to require $(i_0 + 1)(i_1 + 1) \cdots (i_k + 1) = \begin{pmatrix} n+k \\ k \end{pmatrix}$. In complete agreement with arguments made in Section 1, if we choose $n = 1 \cdot 2 \cdot 3 \cdots k$ and then identify i_l with n/l, provided that $0 < l \leq k$, we indeed obtain the required decomposition. This result allows us to think about the partition function for the Veneziano amplitudes in terms of the spin model which is the some kind of reduction of the rigid string model proposed while ago by Polyakov [15]. Unlike his model, our (spin chain) model is exactly solvable. The noticed connection is important in view of the potential physical applications: in Ref. [42], based on our earlier developed variant [16] of the Polyakov rigid string model, interesting applications to QCD were considered. Since the rigid string model proposed by Polyakov is also of Grassmann-type (albeit, apparently, for different reasons) it is of interest to study its reduction to the exactly solvable spin chain models.

7.6. Some physical applications

7.6.1. The Poincaré polynomial and the partition function

Although the arguments presented above are standard, they cannot be used for our immediate tasks. This is so because of the fact that in our case we have to consider the differential forms. Specifically, let us consider a differential form of the type $\mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ with entries f_i, f_2, etc. considered as independent variables. In which case we are dealing with just *one* differential form. Let now $f_0 = z_0^{n_0}$, $f_2 = z_2^{n_2}, \ldots, f_k = z_k^{n_k}$ provided that $n_0 + n_1 + \cdots + n_k = n$. Clearly, under such conditions we shall obtain $\begin{pmatrix} n+k \\ k \end{pmatrix}$ different differential forms which can be looked upon as Plücker embedding of the space $(z_0, \ldots, z_k)$ into the space of differential forms of the type $\mathrm{d}z_0^{n_0} \wedge \mathrm{d}z_1^{n_1} \wedge \cdots \wedge \mathrm{d}z_k^{n_k} \simeq z_0^{n_0} z_1^{n_1} \cdots z_k^{n_k} \frac{\mathrm{d}z_0}{z_0} \wedge \cdots \wedge \frac{\mathrm{d}z_k}{z_k}$. Coordinates $z_0^{n_0-1} z_1^{n_1-1} \cdots z_k^{n_k-1}$ (taken in a prescribed order) can be viewed as Plücker coordinates so that, apparently, we have essentially obtained the Plücker embedding. This is not quite the case yet. To obtain the desired result several additional steps are needed. For instance we can look at projective transformations of the type $z_0 \to lz_0$, etc. Upon such a replacement the combination $\frac{\mathrm{d}z_0}{z_0}$ will stay the same while the combination $z_0^{n_0} z_1^{n_1} \cdots z_k^{n_k}$ will formally change into $l^n z_0^{n_0} z_1^{n_1} \cdots z_k^{n_k}$ and is not invariant with respect to such transformations. To correct the problem we have to divide $z_0^{n_0} z_1^{n_1} \cdots z_k^{n_k}$ by the combination which scales the same way. In view of results of Part I, this will be the Fermat variety $\mathcal{F}(\mathbf{z}) = z_0^n + \cdots + z_k^n$. The Poincaré polynomial obtained in Eq. (6.11) in the limit $z \to 1$ counts the number of such distinct invariant forms. This topic will be discussed further in Section 9 where we define the projective toric varieties and in Part III. The rescaling just described subdivides the complex projective space into equivalence classes. This procedure is essentially equivalent to the Plücker embedding. At the same time, in view of invariance of the combination $\frac{\mathrm{d}z_0}{z_0} \wedge \cdots \wedge \frac{\mathrm{d}z_k}{z_k}$ with respect to scale transformation, one can think about the equivalence classes only between the monomials of the type $z_0^{n_0} z_1^{n_1} \cdots z_k^{n_k}$ and, from this point of view, one obtains the Veronese embedding. Since combinatorially in both cases we have been working with the same objects, not surprisingly, the number of equivalence classes in both cases came out the same. Physically, however, it is more advantageous to use the Poincaré polynomial for the Veronese embedding since, as discussed in Section 1, the Poincaré polynomial for the complex projective space of dimensionality $\mathcal{N}$ (up to numerical prefactor) coincides with the partition function for a particle with spin $\mathcal{N}+1$ placed into constant magnetic field. Such a partition function was obtained by Stone using $N=2$ finite dimensional supersymmetric model. In our case we are interested not only in the partition function counting the number of entries (summands) in the total Veneziano amplitude but also in the possibility of reobtaining these amplitudes with help of this partition function. If this can be achieved, we might consider the Veneziano model as exactly solved. Since we know already that such a partition function up to a constant coincides with the Weyl character formula, we must look for the group-theoretic aspects of results we have just obtained. This is done in the next subsection and the section which follows.

7.6.2. Connections with KP hierarchy

For the sake of space we shall assume familiarity of our readers with the theory of symmetric functions. Excellent exposition can be found in Ref. [43], while the basic facts can be found in Ref. [44]. From these sources, it is known that the Schur functions $\mathbf{s}_\lambda(\mathbf{x})$ play the central role in this theory in view of their mutual orthogonality with respect to carefully chosen scalar product $\langle , \rangle$, i.e. $\langle \mathbf{s}_\lambda, \mathbf{s}_\mu \rangle = \delta_{\lambda,\mu}$.[21] To make a connection with previous discussion, let us consider the following generating function

$$f(\mathbf{z}) = \frac{1}{N!}(z_0 + \cdots + z_k)^N = \sum_{\substack{(n_0, n_1, \ldots, n_k) \\ N = n_0 + n_1 + \cdots + n_k}} \frac{1}{n_0! n_1! \cdots n_k!} z_0^{n_0} \cdots z_k^{n_k} \equiv \sum_{\lambda \vdash N} c_\lambda \mathbf{z}^\lambda, \quad (7.10)$$

[21] We have suppressed the arguments, e.g. $\mathbf{x} = \{x_1, x_2, \ldots x_m\}$, in this product for brevity.

where notations introduced in Section 1 were used.[22] Such an expansion can be considered as some kind of a basis expansion in which the basis vectors belong to the set $\mathbf{z}^\lambda$. From previous subsection we know that such an expansion makes sense since the monomials $\mathbf{z}^\lambda$ represent well defined equivalence classes in complex projective space. In general, however, such monomials are not orthogonal with respect to the scalar product just introduced. Evidently, each of these monomials can be re expanded with help of the Schur polynomials, i.e.

$$\mathbf{z}^\lambda = \sum_{\mu \vdash N} \tilde{c}_{\mu,\lambda} \mathbf{s}_\mu. \tag{7.11}$$

But from the book by Miwa et al, Ref. [45, p. 90], we find out that under such circumstances $\mathbf{z}^\lambda$ represents the tau function of the KP hierarchy. Using this observation, perhaps superimposed with our earlier treatment of the Witten–Konsevich model, Ref. [7], one can develop, in principle, some quantum mechanical model whose partition function will coincide with the Poincaré polynomial discussed earlier. There is much faster way however to arrive at the final destination. It is described in the next section. In the meantime, we would like to provide some qualitative arguments in favour of such an alternative approach. To this purpose, using Eq. (7.11) in (7.10) we can expand $f(\mathbf{z})$ in terms of the orthogonal basis. Suppose now that there is an operator such that $\mathbf{s}_\mu$ is its eigenfunction. Hence, $f(\mathbf{z})$ is also an eigenfunction of such an operator. Since the Hilbert space is finite dimensional in the present case, we may look, using the analogy with the commutator algebra for angular momentum, for some kind of raising and lowering operators. If they indeed exist, then, as for the angular momentum (or spin), there will be the upper and the lower vacuum states. The dimension of the Hilbert state in this case can be determined straightforwardly from the partition function whose Hamiltonian $H = B_z \cdot S_z$ where $\mathbf{B}$ is some external "magnetic" field whose direction, as usual, is chosen to be along the "z-axis". The partition function Z is obtained now in a standard way as

$$Z = \mathrm{tr}(\exp(-\beta H)), \tag{7.12}$$

and coincides with the Weyl character formula. In the limit $B_z \to 0$ one obtains the dimensionality of the Hilbert space as expected. The same result can be obtained differently. For this purpose it is sufficient to choose the Hamiltonian as $H = \mathbf{S}^2 - const$ where the $const$ is determined by some assigned fixed eigenvalue for the square of the total spin (or angular momentum). Under such circumstances taking trace in Eq. (7.12) using eigenfunctions of H will also produce the dimensionality of the Hilbert space. In the next section we shall implement the first procedure explicitly. To do so, we need to describe a model for the complex projective space. It is known that there is number of such models [46]. So, we have to select one of them which is the most convenient for us. It will be used also in Part III.

7.6.3. Description of particular model describing the projective space

To describe the model, we notice that each complex line in $\mathbf{C}^{n+1}$ passing through the origin can be characterized by the unit vector $\omega_\nu^0 = \frac{\omega_\nu}{|\omega_\nu|}$, $\nu = 0, \ldots, n$, so that parametrically it can be represented as $z_\nu = \omega_\nu^0 \xi$ with ξ being some complex parameter. By definition, the projective space $\mathbf{CP}^n$ is made of equivalence classes of points $\mathbf{z} \in \mathbf{C}^{n+1} \backslash \{0\}$ such that $\mathbf{z}' = \lambda \mathbf{z}$ with λ being some nonzero complex number. In the present case such a definition essentially implies $\lambda = \xi$.

[22] We have suppressed an auxiliary variable, say $\mathbf{t}$, which is normally used in generating functions. Clearly, it can be restored whenever it is needed.

Consider now a unit sphere of real dimension $2n+1$ living in $\mathbf{C}^{n+1}$ and centered at the arbitrarily chosen origin. It is characterized by the equation $\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu = 1$. The projective space $\mathbf{CP}^n$ can be realized as the set of points originating from the intersection of such a sphere with the complex line just described. This results in an equation

$$|\xi|^2 \sum_{\nu=0}^{n} |\omega_\nu^0|^2 = 1, \tag{7.13}$$

from which it follows that $|\xi|^2 = 1$ and, hence, $\xi = \mathrm{e}^{\mathrm{i}\varphi}$. Thus, the points of $\mathbf{C}^{n+1}$ can be parametrized by $\omega_\nu^0 \mathrm{e}^{\mathrm{i}\varphi}$ with ω_ν^0 being some nonnegative numbers subject to the constraint $\sum_{\nu=0}^{n} (\omega_\nu^0)^2 = 1$. For the purposes of our discussion sometimes it will be convenient to redefine $(\omega_\nu^0)^2$ as t_ν so that in terms of such variables the constraint describes a simplex instead of a sphere.[23] Two points of $\mathbf{C}^{n+1}$ differing by their phase factors belong to the same equivalence class so that the whole $\mathbf{C}^{n+1}$ is divided into equivalence classes which are labeled by $\omega_\nu^{0'} s$. Because $\omega_\nu^{0'} s$ are subject to the constraint, the dimensionality of thus formed projective space $\mathbf{CP}^n$ is n. Evidently, earlier discussed quotient $z_0^{n_0} z_1^{n_1} \cdots z_k^{n_k} / \mathcal{F}(\mathbf{z})$ will remain invariant with respect to such parametrization. This fact was crucial for reconstruction of the Veneziano amplitudes from periods of the Fermat varieties. Further implications of this invariance will be discussed in Section 9 and in Part III.

Remark 7.2. From Appendix A, part d, it follows that the action of elements of pseudo-reflection groups on some prescribed positive definite Hermitian form leaves this form invariant. This fact can be taken as defining property of the pseudo-reflection groups. The model of projective space just described is compatible with actions of elements of the pseudo-reflection groups. One can say as well that such groups can exist only in certain spaces thus reflecting properties of these spaces. This fact will be analyzed further in Section 9 (also in connection with Remark 7.4). In fact, all results of previous sections are just consequences of this observation.

Remark 7.3. From the previous remark it follows that the action of group elements is taking place component wise. This fact will be used in the next section.

Remark 7.4. The quadratic form $\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu = 1$ can be extended to $\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu = z_{n+1}\bar{z}_{n+1}$. For $z_{n+1}\bar{z}_{n+1} \neq 0$ this form can be reduced back to the original. However, if we keep it with this extra term, it becomes an invariant for groups of isometries of the complex hyperbolic space. Such a space was analyzed thoroughly by Goldman, Ref. [47]. Its profound physical significance will be discussed in Section 9.

8. Exact solution of the Veneziano model

8.1. From Witten to Lefschetz

We begin with the following observations. First, given that Veneziano (and Veneziano-like) amplitudes are periods of the Fermat (hyper)surfaces, the associated with such periods invariant differential forms are living in the complex projective space. The theorem by Kodaira asserts that a compact complex manifold X is projective algebraic if it is a Hodge manifold. For the sake of space, we refer our readers to the monograph by Wells, Ref. [48], for more information. We shall use this reference extensively in what follows.

[23] The simplex was used already in Part I. It will be further discussed in Section 9 and in Part III.

Second, the Theorem 2.1.7 of this reference states that under conditions of Kodaira's theorem the manifold X can be embedded into complex Grassmannian so that the differential forms will be also living in the Grassmannian as discussed earlier.

Third, for a complex Hermitian manifold X let $\mathcal{E}^{p+q}(X)$ denote the set of complex-valued differential forms (sections) of the type (p, q), $p + q = r$, living on X. The Hodge decomposition insures that $\mathcal{E}^r(X) = \sum_{p+q=r} \mathcal{E}^{p+q}(X)$. The Dolbeault operators ∂ and $\bar{\partial}$ act on $\mathcal{E}^{p+q}(X)$ as follows $\partial : \mathcal{E}^{p+q}(X) \to \mathcal{E}^{p+1,q}(X)$ and $\bar{\partial} : \mathcal{E}^{p+q}(X) \to \mathcal{E}^{p,q+1}(X)$. The exterior derivative operator is defined as $d = \partial + \bar{\partial}$. Let now $\varphi_p, \psi_p \in \mathcal{E}^p(X)$ where $\mathcal{E}^p(X)$ belongs to the elliptic complex $\mathcal{E}^*(X) = \bigoplus_{p=0}^{r} \mathcal{E}^p(X)$ of differential forms on X forming a complex vector space of dimension $r + 1$.[24] By analogy with traditional quantum mechanics one can define (using Dirac's notations) the inner product

$$\langle \varphi_p | \psi_p \rangle = \int_M \varphi_q \wedge *\bar{\psi}_p, \tag{8.1}$$

where the bar means the complex conjugation and the star $*$ means the Hodge conjugation as usual. The period integrals, e.g. those for the Veneziano-like amplitudes, are expressible through such inner products [48]. Fortunately, such a product possesses properties typical for the finite dimensional quantum mechanical Hilbert spaces. In particular,

$$\langle \varphi_p | \psi_q \rangle = C\delta_{p,q} \quad \text{and} \quad \langle \varphi_p | \varphi_p \rangle > 0, \tag{8.2}$$

where C is some known positive constant.

Fourth, with respect to such defined scalar product it is possible to define all conjugate operators, e.g. d^*, etc. and, most importantly, the Laplacians

$$\Delta = dd^* + d^*d, \qquad \Box = \partial\partial^* + \partial^*\partial, \qquad \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{8.3}$$

All this was known to mathematicians before Witten's work [49]. The unexpected twist occurred when Witten suggested to extend the notion of the exterior derivative d. Within the de Rham picture (valid for both real and complex manifolds) let M be a compact Riemannian manifold and K be the Killing vector field which is just one of the generators of isometry of M. Then Witten suggested to replace the exterior derivative operator d by the extended operator

$$d_s = d + si(K). \tag{8.4}$$

Here s is real nonzero parameter conveniently chosen. Witten argued that one can construct the modified Laplacian by replacing conventional Δ given in Eq. (8.3) by $\Delta_s = d_s d_s^* + d_s^* d_s$. This is possible if and only if $d_s^2 = d_s^{*2} = 0$ or, since $d_s^2 = s\mathcal{L}(K)$, where $\mathcal{L}(K)$ the Lie derivative along the field K, acting on the corresponding differential form, vanishes. The details are beautifully explained in the much earlier paper by Frankel [50] to be discussed in Part III. Atiyah and Bott, Ref. [22], observed that replacement of the operator d by d_s causes replacement of the de Rham cohomology by the equivariant cohomology. This topic is mentioned in Ref. [6] and will be discussed in more detail in Part III in connection with designing of the symplectic model reproducing the Veneziano amplitudes. In this work we shall use more traditional methods however based on Eq. (8.3).

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[24] Actually, one should consider instead a more complicated object $\mathcal{E}^*(X, E)$ of differential forms with coefficients in E where E is the Hermitian vector bundle over X. This would lead us to the discussion involving the sheaf theory, Čhech cohomology, etc. These are beautifully explained in Ref. [46]. Since the final results which we obtain are not going to be affected, we are not going to complicate matters by these intricacies.

Looking at these equations and following Refs. [26,51,52] we define the (Dirac) operator $\acute{\partial} = \bar{\partial} + \bar{\partial}^*$ and its adjoint with respect to scalar product, Eq. (8.2). Then, use of the above references allows us to determine the dimension Q of the quantum Hilbert space for which the scalar product, Eq. (8.2), was defined. It is given by

$$Q = \ker \acute{\partial} - co, \ker \acute{\partial}^* = Q^+ - Q^-. \tag{8.5}$$

We would like to arrive at the same result differently using earlier introduced partition function, Eq. (7.12). To this purpose we notice that according to Theorem 4.7. in the book by Wells [48] we have $\Delta = 2\Box = 2\bar{\Box}$ with respect to the Kähler metric on X. Next, according to the Corollary 4.1.1. of the same reference Δ commutes with $d, d^*, \partial, \partial^*, \bar{\partial}$ and $\bar{\partial}^*$. From these facts it follows immediately that if we, in accord with Witten, choose Δ as our Hamiltonian, then the supercharges can be selected as $Q^+ = d + d^*$ and $Q^- = i(d - d^*)$. Evidently, this is not the only choice as Witten also indicates. If the Hamiltonian H is acting in *finite* dimensional Hilbert space one may require axiomatically that: (a) there is a vacuum state (or states) $|\alpha\rangle$ such that $H|\alpha\rangle = 0$ (i.e. this state is the harmonic differential form) and $Q^+|\alpha\rangle = Q^-|\alpha\rangle = 0$. This implies, of course, that $[H, Q^+] = [H, Q^-] = 0$. Finally, once again, following Witten, we require that $(Q^+)^2 = (Q^-)^2 = H$. Then, the equivariant extension, Eq. (8.4), leads to $(Q_s^+)^2 = H + 2\, is\, \mathcal{L}\,(K)$. Fortunately, we can avoid this extension by noticing that the above supersymmetry algebra can be extended. This can be done with help of the Lefschetz isomorphism theorem whose exact formulation is given as Theorem 3.1.2 in Wells, Ref. [48]. We shall only use some parts of this theorem in our work. In particular, using notations of Ref. [48], we introduce the operator L commuting with Δ and its adjoint $L^* \equiv \Lambda$. It can be shown [48, p. 159], that $L^* = w * L*$ where, as before, $*$ denotes the Hodge star operator and the operator w can be formally defined through the relation $** = w$ [48, p. 156]. From these definitions it should be clear that L^* also commutes with Δ on the space of harmonic differential forms (in accord with page 195 of [48]).

As part of preparation for proving of the Lefschetz isomorphism theorem, it can be shown [48], that

$$[\Lambda, L] = B \quad \text{and} \quad [B, \Lambda] = 2\Lambda, \qquad [B, L] = -2L. \tag{8.6}$$

This commutator algebra (up to a constant) coincides with the $sl_2(\mathbf{C})$ Lie algebra given in the canonical form, e.g. see Ref. [53, p. 37], as follows

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{8.7}$$

Comparison between the above two expressions leads to the isomorphism of Lie algebras, i.e. the operators h, f and e act on the vector space $\{v\}$ to be described below while the operators Λ, L and B obeying the same commutation relations act on the space of differential forms.

Remark 8.1. For such an isomorphism to exist the elliptic complex should be finite dimensional. This requirement of finite dimensionality comes from important result by Serre to be described in the next subsection

8.2. From Lefschetz to Veneziano via Serre and Ginzburg

Now we would like to recall, e.g. Ref. [53], p. 25, that all semisimple Lie algebras are made of copies of $sl_2(\mathbf{C})$. Assuming our readers familiarity with the Lie algebras and, in particular, with semisimple Lie algebras, we would like now to adopt the Lefschetz correspondence to our needs. In particular, let $z_1, \ldots, z_n$ denote a basis of the vector space V in $\mathbf{C}^n$. In terms of this basis consider a polynomial $f(\mathbf{z})$ given by

$$f(\mathbf{z}) = f(z_1, \ldots, z_n) = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} \equiv \sum_{\mathbf{i}} \lambda_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n}, \quad (\lambda_{\mathbf{i}}, z_m^{i_m} \in \mathbf{C}, 1 \leq m \leq n). \tag{8.8}$$

Let $\mathbf{z}$ be treated as a column vector (or, better, as a set of column vectors, e.g. see Section 7.1). Then, by definition (e.g. see Ref. [54] compatible with earlier made Remark 7.3) we obtain, $M \circ f(\mathbf{z}) = f(M\mathbf{z}) \equiv f(Mz_1, \ldots, Mz_n)$, where $M \subset G$. Here G belongs to some matrix group. In particular, let $M \subset sl_2(\mathbf{C})$, then following Dixmier [55, Chapter 8], we introduce operators $h = \sum_{\alpha=1}^{n} a_\alpha h_\alpha$, $e = \sum_{\alpha=1}^{n} b_\alpha e_\alpha$, $f = \sum_{\alpha=1}^{n} c_\alpha f_\alpha$. Provided that the constants are subject to the constraint: $b_\alpha c_\alpha = a_\alpha$, the commutation relations between the operators h, e and f are *exactly the same* as respectively for B, Λ and L. To avoid unnecessary complications, we choose $a_\alpha = b_\alpha = c_\alpha = 1$.

Next, following Serre [56, Chapter 4], we need to introduce the *primitive* vector (or element). This is the vector v such that $hv = \lambda v$ but $ev = 0$. The number λ is the weight of the module $V^\lambda = \{v \in V | hv = \lambda v\}$. If the vector space is *finite dimensional*, then $V = \sum_\lambda V^\lambda$. Moreover, only if V^λ is finite dimensional it is straightforward to prove that the primitive element *does* exist in accord with Remark 8.1. The proof is based on observation that if x is the eigenvector of h with weight λ, then ex is also an eigenvector of h with eigenvalue $\lambda - 2$, etc. Moreover, from the book by Kac [57, Chapter 3], it follows that if λ is the weight of V, then $\lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ is also the weight with the same multiplicity. Since according to Eq. (A.2) of Appendix A $\langle \lambda, \alpha_i^\vee \rangle \in \mathbf{Z}$, Kac introduces another module: $U = \sum_{k \in \mathbf{Z}} V^{\lambda + k\alpha_i}$. Such a module is finite for finite reflection groups and is infinite for the affine reflection groups. We would like to argue that for our purposes, in view of Theorem 2.2 by Solomon it is sufficient to use only finite reflection (or pseudo-reflection) groups.

Remark 8.2. It should be remembered at this point that in Solomon's theorem the requirement of finiteness of the (pseudo)reflection group is stated explicitly.

Remark 8.3. From the book by Kac, it should be clear that the infinite dimensional version of the module U straightforwardly leads to all known string-theoretic results. Development of connections with KP hierarchy discussed in Section 7.2 also ultimately leads to the conventional string-theoretic formulations. In the case of CFT this is essential and will be explained further in Section 9 and in Part IV, but for calculation of the Veneziano-like amplitudes this is *not* essential. By accepting the traditional option we loose connections with the Lefschetz isomorphism theorem (relying heavily on the existence of primitive elements) and with the Hodge theory in its traditional form. The infinite dimensional extensions of the Hodge-de Rham theory involving loop groups, etc. relevant for the CFT can be found in Ref. [58,59]. Fortunately, they are not needed for the purposes of this work. Hence, below we work only with finite dimensional spaces.

In particular, let now v be a primitive element of weight λ. Then, following Serre, we let $v_n = \frac{1}{n!} e^n v$ for $n \geq 0$ and $v_{-1} = 0$, so that

$$hv_n = (\lambda - 2n)v_n, \qquad ev_n = (n+1)v_{n+1}, \qquad fv_n = (\lambda - n + 1)v_{n-1}. \tag{8.9}$$

Clearly, the operators e and f are the creation and the annihilation operators according to existing in physics terminology while the vector v can be interpreted as the vacuum state vector. The question arises: how this vector is related to earlier introduced vector $|\alpha\rangle$? Before providing the answer to this question we need, following Serre, to settle the related issue. In particular, we can either:

(a) assume that for all $n \geq 0$ the first of Eq. (8.9) has solutions and all vectors $v, v_1, v_2, \ldots$, are linearly independent or (b) beginning from some $m + 1 \geq 0$, all vectors v_n are zero, i.e. $v_m \neq 0$ but $v_{m+1} = 0$. The first option leads to the infinite dimensional representations associated with affine Kac-Moody algebras just mentioned. The second option leads to the finite dimensional representations and to the requirement $\lambda = m$ with m being an integer. We shall adjust this integer to our needs shortly below. In the meantime, following Serre, this observation can be exploited further thus leading us to the crucial physical identifications. Serre observes that with respect to $n = 0$ Eq. (8.9) possess a ("super") symmetry. That is the linear mappings

$$e^m : V^m \to V^{-m} \quad \text{and} \quad f^m : V^{-m} \to V^m, \tag{8.10}$$

are isomorphisms and the dimensionality of V^m and V^{-m} are the same. Serre provides an operator (the analog of Witten's F operator) $\theta = \exp(f)\exp(e)\exp(-f)$ such that $\theta \cdot f = -e \cdot \theta, \theta \cdot e = -\theta \cdot f$ and $\theta \cdot h = -h \cdot \theta$. In view of such an operator, it is convenient to redefine the h operator: $h \to \hat{h} = h - \lambda$. Then, for such redefined operator the vacuum state is just v. Since both L and $L^* = \Lambda$ commute with the supersymmetric Hamiltonian H and, because of the group isomorphism, we conclude that the vacuum state $|\alpha\rangle$ for H corresponds to the primitive state vector v, moreover, $-m \leq n \leq m$ in Eq. (8.9). Now we are ready to apply yet another isomorphism following Ginzburg, Ref. [21, pp. 205-206].[25] To this purpose we make the following identification

$$e_i \to t_{i+1}\frac{\partial}{\partial t_i}, \qquad f_i \to t_i\frac{\partial}{\partial t_{i+1}}, \qquad h_i \to t_i\frac{\partial}{\partial t_i} + 2\left(t_{i+1}\frac{\partial}{\partial t_{i+1}} - t_i\frac{\partial}{\partial t_i}\right), \tag{8.11}$$

$i = 0, \ldots, m$. Such operators are acting on the vector space made of monomials of the type

$$v_n \to \mathcal{F}_N = \frac{1}{n_0!n_2!\cdots n_k!}t_0^{n_0}\cdots t_k^{n_k}, \tag{8.12}$$

where $n_0 + \cdots + n_k = N$. This result is in accord with. Eq. (7.10). Moreover, now we have analogs of Eq. (8.9). These are given by

$$\begin{aligned} &h_i * \mathcal{F}_n(i) = 2(n_{i+1} - n_i)\mathcal{F}_n(i), \qquad e_i * \mathcal{F}_n(i) = 2n_i\mathcal{F}_n(i+1),\\ &f_i * \mathcal{F}_n(i) = 2n_{m+1}\mathcal{F}_n(i-1), \end{aligned} \tag{8.13}$$

where $\mathcal{F}_n(i)$ is a part of the wave function relevant to action of operators e_i, f_i, h_i. Clearly, at this point one should make the following identifications: $m(i) - 2n(i) = 2(n_{i+1} - n_i)$, $2n_i = n(i) + 1$ and $m(i) - n(i) + 1 = 2n_{i+1}$ in order to be consistent with Eq. (8.9). Next, we define the total Hamiltonian: $h = \sum_{i=0}^{k} h_i$ [26] and redefine individual Hamiltonians as described above. This causes $m(i)$ to be effectively zero in the above equations. The operators $\frac{\partial}{\partial t_i}$ act on the total set of monomials $1, t, \frac{1}{2!}t^2, \ldots, \frac{1}{N!}t^N$. Such monomials are forming the basis of vector space analogous to v_n. This leads to identification of $m(i)$ with $N\ \forall i$. Incidentally, N is the total energy according to Veneziano condition, Eq. (1.4). Based on these remarks let us consider an action of such redefined

[25] Unfortunately, the original source contains very minor mistakes (misprints). These are easily correctable. The corrected results are given in the text.

[26] In accord with general rules of construction of the Lie algebras out of copies of $sl_2(\mathbf{C})$ thus designed Hamiltonian represents the standard action of $sl_k(\mathbf{C})$ on the vector space made out of monomials, Eq. (8.12).

total Hamiltonian on the individual wave function $\hat{\mathcal{F}}_N$ of the type given by Eq. (8.12). Using Eq. (8.13) we obtain,

$$h\hat{\mathcal{F}}_N = 2(n_k - n_0)\hat{\mathcal{F}}_N. \tag{8.14}$$

Since $0 \leq n_i \leq N\ \forall i$ we obtain: $0 \leq |n_k - n_0| \leq N$. In view of this Eq. (8.14) becomes very much analogous to the equation for the z-component of the angular momentum (or spin) of magnitude N. Unlike the case of angular momentum, here there is an additional degeneracy which we would like to describe now. To this purpose we need to describe in some detail the wave functions entering Eq. (8.14).

Firstly, we notice that the operators introduced in Eq. (8.11) do not change the total power of monomials of the type given in Eq. (8.12). This is in accord with the requirement that for such monomials the constraint $n_0 + \cdot + n_k = N$ should always hold. Next, for completeness of our presentation we would like to restore the subtracted term in the total Hamiltonian. Then, its action on the monomial $\frac{1}{N!}t_0^N$ produces an eigenvalue $-N$. Furthermore, consider now another monomial $\frac{1}{(N-1)!}t_0^{N-1}t_1$. The action of a Hamiltonian on such a monomial will produce an eigenvalue $-N+2$ as required [56]. But the same eigenvalue will be produced also by the monomials of the type $\frac{1}{(N-1)!}t_0^{N-1}t_i$, $i = 2, 3, \ldots, k$. Hence we have obtained a degeneracy. The next generation of wave functions can be constructed as follows $\frac{1}{(N-2)!}t_0^{N-2}t_it_j$, $\frac{1}{(N-3)!}t_0^{N-3}\frac{1}{2!}t_i^2t_j$, etc. This process will end when we shall reach the situation when we would have $t_0^0 \ldots$ Such wave function will have an eigenvalue zero. Next, we can obtain another series of wave functions which begins with $\frac{1}{N!}t_k^N$. To construct this series we have to switch signs in the corresponding equations according to rules implied by Eq. (8.10). These two series exhaust all the combinations satisfying $n_0 + \cdots + n_k = N$. Clearly, the number of such combinations $\mathcal{N} = \begin{pmatrix} N+k \\ k \end{pmatrix}$ in accord with Section 1. Since all wave functions given by Eq. (8.12) possess the same energy N, the partition function, Eq. (7.12), in the limit $\beta \to 0$ reproduces $\mathcal{N}$ as required. Since thus constructed wave functions are in one-to one relation with the corresponding Veneziano amplitudes, obtained results provide a complete solution of the Veneziano model.

8.3. Connections with chaotic dynamical systems and problem of zeros of the Riemann zeta function

Some of our readers may ask at this point the following question: all this is fine but what kind of physics the Hamiltonian introduced in Eq. (8.11) represents? We would like to address this important issue now. First, even if this Hamiltonian would be a formality, because of the Lefschets isomorphism theorem we can always go back to the traditional supersymmetric formulation of the problem. Such a formulation, although physically useful, leaves certain aspects of the problem undetected. This is especially true in the present case since by using the supersymmetric formulation the fact that four-particle Veneziano amplitude can be equivalently presented as the product of Riemann zeta functions (e.g. see Eq. (1.12) of Part I) seems only as a curiosity. This curiosity happens to be intrinsically related to Hamiltonians of the type given by Eq. (8.11). The simplest Hamiltonian of this type is $H = xp$. It was recently considered by Berry and Keating, Ref. [60], and, more comprehensively, in Ref. [61]. These authors notice that at the classical level the system described by such Hamiltonian has a hyperbolic point at the origin of the (x, p) phase space plane. The trajectories $x(t) = x(0)exp(t)$ and $p(t) = p(0)exp(-t)$ are uniformly unstable with stretching in x and contraction in p. The motion has the desired lack of time reversal symmetry so that the orbit cannot be retracted.[27] Quantization of classically chaotic systems is currently in the focus of attention in physics literature [63] and in the case of the system just described can be directly connected with zeros of the Riemann zeta function and, hence, with the Riemann hypothesis about these zeros [60]. Okubo [64] noticed the Lorentz invariance of two dimensional Hamiltonians of the type given by Eq. (8.11) and made a conjecture about some intrinsic connections between the Lorentz invariance and the Riemann hypothesis. The Lorentz invariance of $N = 2$ supersymmetric quantum mechanics was noticed already in the seminal paper by Witten, Ref. [49, p. 662]. No connections with the Riemann hypothesis were made however in his paper. We shall discuss further the issues related to the Lorentz invariance in Section 9, mainly elaborating on Remark 7.4 made earlier.

In mathematics literature, the ergodic properties of the dynamical systems associated with semisimple Lie groups and algebras have received considerable attention recently, e.g. see monograph by Feres, Ref. [65]. In our opinion, the dynamical issues in the present case intrinsically are of the number-theoretic nature. The number-theoretic aspects of the Veneziano amplitudes discussed in our earlier publication, Ref. [10], provide a natural link between dynamics, Rieamnn's zeta function and, more general, L-functions. For the sake of space, we refer our readers to just mentioned literature containing, in addition, a large number of relevant references of major importance.

9. The theorem by Serre and its physical significance

9.1. Statement of the theorem and physically motivated proof

In the previous sections we repeatedly mentioned the theorem by Serre. In this section we would like to state the theorem explicitly, to explain using physical arguments its proof, and to discuss some physical consequences of this theorem not mentioned thus far.

Before stating the theorem, we need to recall that any linear algebraic group G is isomorphic to a closed subgroup of $GL_n(V, K)$ acting on a vector space V of dimension $n \geq 1$ by matrices M whose entries belong to any closed number field K such as $\mathbf{C}$ or p-adic [66]. With such an observation, we are ready to formulate the theorem, e.g see Bourbaki, Ref. [17], Chapter 5, paragraph 5 (problem set #8).

Theorem 9.1 (Serre [67]). *Let V be a vector space of dimension n over the field K and let $S(V)^G$ be a graded ring of invariants of the group G acting on symmetric algebra $S(V)$ (defined in* Section 2*). Then $S(V)^G$ is a polynomial algebra if and only if G is finite group generated by pseudo-reflections.*

Remark 9.2. It is important that the theorem by Serre involves only finite pseudo-reflection groups. This requirement is consistent with earlier stated Theorem 5.1 by McMullen. Below we shall mention the conditions under which it should be amended in order to reproduce the results of CFT.

Although the proof of this theorem can be found in many places, e.g. see Refs. [68,69] or the original paper by Serre, Ref. [67], we would like to provide arguments leading to a physically motivated proof.

[27] In Ref. [62] it is demonstrated that the Hamiltonian $H = xp$ is canonically equivalent to the Hamiltonian for the "inverted" harmonic oscillator: $H = \frac{1}{2}(P^2 - Q^2)$, obtained upon the symplectic rotation of the type: $x = \frac{P+Q}{\sqrt{2}}$, $p = \frac{P-Q}{\sqrt{2}}$.

To begin, let us recall that in Section 4 we defined $x \in V^G$ as a group invariant (for some group G) if $Av(x) = x$. The averaging (over group G) operator Av was defined in Eq. (4.5). Following Stanley, Ref. [54], we would like now to provide few additional details. For instance, taking into account Eq. (8.8) and discussion which follows this equation the polynomial ring $A[\mathbf{z}]$, $\mathbf{z} \in C^n$, contains a subring $S(V)^G$ of invariants defined by

$$S(V)^G = \{f(\mathbf{z}) \in A[\mathbf{z}] : M \circ f(\mathbf{z}) = f(M\mathbf{z}) = f(\mathbf{z}) \quad \forall M \in G\}. \tag{9.1}$$

If $X(G)$ is the set of all irreducible (complex, in general) characters of G, then $A[\mathbf{z}]$ can be decomposed into direct sum as follows: $A[\mathbf{z}] = \coprod_\chi S(V)^G_\chi$, where the condition $f(\mathbf{z}) \in S(V)^G_\chi$ means that

$$S(V)^G_\chi = \{f \in A[\mathbf{z}] : M \circ f(\mathbf{z}) = \chi(M) f(\mathbf{z}) \quad \forall M \in G \quad \text{and} \quad \chi(M) \in X(G)\}. \tag{9.2}$$

From this it follows, that earlier defined $S(V)^G = S(V)^G_\varepsilon$ where ε denotes the trivial character.

Let $n = \dim V$ be the *degree* of G while $|G|$ be its *cardinality*. Emmy Noether [70] proved the following theorem

Theorem 9.3 (Noether). *Let G be of cardinality $|G|$ and n is its degree, then $S(V)^G$ is generated as an algebra over* $\mathbf{C}$ *by no more than* $\begin{pmatrix} |G| + n \\ n \end{pmatrix}$ *homogenous invariants of degree not exceeding* $|G|$.

The results of previous sections are in complete accord with this theorem. However, now we are in the position to develop some refinements. This can be accomplished in several steps. For instance, to put the results of Section 7.6.3 into proper perspective we need to introduce the following

Definition 9.4. The set $T := (\mathbf{C}\backslash 0)^n =: (\mathbf{C}^*)^n$ is called a *complex algebraic torus*.

Since each $z \in \mathbf{C}^*$ can be written as $z = r \exp(i\theta)$ so that for $r > 0$ the fiber: $\{z \in \mathbf{C}^* \| z| = r\}$ is a circle of radius r, we can represent T as the product $(R_{>0})^n \times (S^1)^n$. The product of n circles $(S^1)^n$ is the deformation retract of T. It is indeed a topological torus. Following Fulton [24], we are going to call it a *compact torus* S_n. Hence, the algebraic torus is a product of a compact torus and a vector space. This circumstance is helpful since whatever we can prove for the deformation retract can be extended to the whole torus T. As an illustration relevant to our calculations of the Veneziano amplitude made in Part I, and to discussions we had in Section 7, we would like following Fulton, Ref. [24], to consider a deformation retract of the complex projective space $\mathbf{CP}^n$. Such a retraction is achieved by using the map

$$\tau : \mathbf{CP}^n \to \mathbf{P}^n_{\geq} = \mathbf{R}^{n+1}_{\geq} \setminus \{0\}/\mathbf{R}^+$$

or, explicitly,

$$\tau : (z_0, \ldots, z_n) \mapsto \frac{1}{\sum_i |z_i|}(|z_0|, \ldots, |z_n|) = (t_0, \ldots, t_n), \quad t_i \geq 0. \tag{9.3}$$

The mapping τ is onto the standard n-simplex: $t_i \geq 0$, $t_0 + \cdots + t_n = 1$.

Since we are interested in the torus action on the algebraic variety the above constructed deformation retract simplifies matters considerably. We had a chance to see these simplifications in Part I when we performed our calculations of the Veneziano amplitudes. Now, however, we would like to consider more general cases. To this purpose we provide the following

Definition 9.5. An *affine* algebraic variety $V \in \mathbf{C}^n$ is the set of zeros of the collection of polynomials from the ring $A[\mathbf{z}]$.

According to the famous Hilbert's Nullstellensatz a collection of such polynomials is *finite* and forms the set $I(\mathbf{z}) := \{f \in A[\mathbf{z}], f(\mathbf{z}) = 0\}$ of maximal ideals usually denoted as Spec $A[\mathbf{z}]$. Using this fact, we provide the following formal definition.

Definition 9.6. The zero set of a *single* function which belongs to $I(\mathbf{z})$ is called *algebraic hypersurface*. Accordingly, the set $I(\mathbf{z})$ corresponds to *intersection* of a finite number of hypersurfaces.[28]

Being armed with such results, we would like to construct the affine toric variety and consider the torus action on such a variety. This is accomplished in several steps. First, instead of considering the set of Laurent monomials of the type $\lambda\mathbf{z}^{\alpha} \equiv \lambda z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in A[\mathbf{z}]$, we would like to consider a subset made of monic monomials, i.e. those with $\lambda = 1$. Such a subset forms a subring with respect to the usual multiplication and addition. The crucial step forward is to assume that for such monomials the exponent $\alpha \in S_\sigma$. The monoid S_σ will be defined momentarily. This fact allows us to define the mapping

$$u_i := z^{a_i}, \tag{9.4}$$

with a_i being one of the generators of the monoid S_σ and $z \in \mathbf{C}$. The monoid S_σ can be defined now as follows.

Definition 9.7. *A semi-group S* that is a non-empty set with associative operation is called *monoid* if it is commutative, satisfies cancellation law (i.e. $s + x = t + x$ implies $s = t$ *for all* $s, t, x \in S$) and has zero element (i.e. $s + 0 = s$, $s \in S$). *A monoid S_σ is finitely generated* if there exist some set of $a_1, \ldots, a_k \in S$, called *generators*, such that

$$S_\sigma = \mathbf{Z}_{\geq 0}a_1 + \cdots + \mathbf{Z}_{\geq 0}a_k. \tag{9.5}$$

Based on this, we can make a *crucial observation*: the mapping given by Eq. (9.4) provides an isomorphism between the *additive* group of exponents a_i and the *multiplicative* group of monic Laurent polynomials. Next, we recall that the function ϕ is considered to be *quasi homogenous* of degree d with exponents $l_1, \ldots, l_n$ if

$$\phi(\lambda^{l_1}x_1, \ldots, \lambda^{l_n}x_n) = \lambda^d\phi(x_1, \ldots, x_n), \tag{9.6}$$

provided that $\lambda \in \mathbf{C}^*$. Applying this result to $z^{\mathbf{a}} \equiv z_1^{a_1} \cdots z_n^{a_n}$ we obtain the Veneziano-like equation

$$\sum_j (l_j)_i a_j = d_i. \tag{9.7}$$

Clearly, if the index i is numbering different monomials, then the sum in Eq. (9.7) (equal to d_i) belongs to the monoid S_σ. The same result can be achieved if instead we would consider the products of the type $u_1^{l_1} \cdots u_n^{l_n}$ and rescale all z_i's by *the same* factor λ. Actually, Eq. (9.7) should

[28] In Section 3 (and in Part I) the pole $Q = 0$ of the period integral, Eq. (3.3), defines the algebraic hypersurface. For the Veneziano amplitude it is the Fermat hypersurface. More generally, earlier studies of the scattering amplitudes using Feynman's diagrammatic rules [71] produced similar types of period integrals. Typically, the denominator Q for such integrals is a product of several algebraic functions. Therefore, it should be clear that methods developed for calculation of the Veneziano amplitudes are fully consistent with earlier studies of scattering amplitudes.

be understood as a scalar product between $(l_j)_i$'s (living in the space *dual* to a_j's) and the a_j's. It is convenient at this point to define a cone.

Definition 9.8. A *convex polyhedral* cone σ is a set

$$\sigma = \left\{ \sum_{i=1}^{k} r_i a_i, r_i \geq 0 \right\}. \tag{9.8}$$

Remark 9.9. In the case when earlier introduced generators $a_1, \ldots, a_k$ are considered as the basis of a vector space V, the definitions of S_σ and σ *describe the same object*. We shall always refer to it as a cone. In this case the dual cone $\sigma^\vee$ is defined by

Definition 9.10.

$$\sigma^\vee = \{\mathbf{l} \in V^* : \langle \mathbf{l}, \mathbf{y} \rangle \geq 0 \quad \forall \mathbf{y} \in \sigma\}. \tag{9.9}$$

It explains why the set of $(l_j)_i$'s "lives" in the space dual to that for a_j's. Next, in view of the results just described, we can rewrite Eq. (8.8) as

$$f(\mathbf{z}) = \sum_{\mathbf{a} \in S_\sigma} \lambda_{\mathbf{a}} \mathbf{z}^{\mathbf{a}} = \sum_{\mathbf{l}} \lambda_{\mathbf{l}} \mathbf{u}^{\mathbf{l}}. \tag{9.10}$$

As before, these polynomials form a polynomial ring. The ideal for this ring can be constructed based on observation that for the fixed d_i and the assigned set of cone generators a_i' there is more than one set of generators for the dual cone. This redundancy produces relations of the type

$$u_1^{l_1} \cdots u_k^{l_k} = u_1^{\tilde{l}_1} \cdots u_k^{\tilde{l}_k}. \tag{9.11}$$

If now we require $u_i \in \mathbf{C}_i$, then the above equation belongs to the ideal $I(\mathbf{z})$ of the above polynomial ring. In accord with Definition 9.6, Eq. (9.11) represents a hypersurface. Naturally, the ideal $I(\mathbf{z})$ represents the intersection of these hypersurfaces. The *affine toric* variety $X_{\sigma^\vee}$ is made out of the hypersurfaces which belong to $I(\mathbf{z})$. The generators $\{u_1, \ldots, u_k\} \in \mathbf{C}^k$ are coordinates for $X_{\sigma^\vee}$. They represent the same point in $X_{\sigma^\vee}$ if and only if $\mathbf{u}^{\mathbf{l}} = \mathbf{u}^{\tilde{\mathbf{l}}}$. Thus formed toric variety corresponds to just one (dual) cone. The set of cones having a common origin can be assembled into a fan [24,25]. The fan is *complete* if it spans the k-dimensional vector space. In fact, using results of Appendix A, part A.1, we notice that there is one-to-one correspondence between the cones and the chambers defined in Appendix A. From chambers one can construct a gallery and, hence, a building. So that a complete fan is essentially a building. The information leading to the design of a particular building can be thus used for construction of a toric variety from the set Σ (a complete fan) of affine toric varieties. To do so, one needs the set of gluing maps $\{\Psi_{\sigma^\vee \sigma^\vee}\}$. Thus, we obtain the following

Definition 9.11. Let Σ be a complete fan and $\coprod_{\sigma^\vee \in \Sigma} X_{\sigma^\vee}$ be the disjoint union of affine toric varieties. Then, using the set of gluing maps $\{\Psi_{\sigma^\vee \check{\sigma}^\vee}\}$ such that each of them identifies two points $x \in X_{\sigma^\vee}$ and $\check{x} \in X_{\check{\sigma}^\vee}$ on respective affine varieties, one obtains the toric variety X_Σ determined by the fan Σ.

Thus constructed variety X_Σ may contain singularities. This should be obvious just by looking at Eq. (9.11). There is a procedure of desingularization described, for example, in Ref. [24] which we are going to by-pass. This is permissible in view of the results we have discussed thus far. Clearly, if physically needed, such more complicated varieties can be studied as well. Obtained results allow us to introduce the following

Definition 9.12. The *torus action* is a continuous map: $T \times X_{\Sigma} \to X_{\Sigma}$ such that for each affine variety corresponding to the dual cone it is given by

$$T \times X_{\sigma^{\vee}} \to X_{\sigma^{\vee}}, \qquad (t, x) \mapsto tx := (t^{a_1} x_1, \ldots, t^{a_k} x_k). \tag{9.12}$$

Naturally, such an action should be compatible with the gluing maps thus extending it from one cone (chamber) to the entire variety X_{Σ} (building). The compatibility is easy to enforce since for each of Eq. (9.11) multiplication by t-factors will not affect the solutions set. This can be formally stated as follows. Let $\Psi : X_{\Sigma} \to X_{\tilde{\Sigma}}$ be a map and $\alpha : T \to T'$ a homomorphism, then the map Ψ is called *equivariant* if it obeys the following rule compatible with earlier defined Eq. (9.2):

$$\Psi(cx) = \chi(c)\Psi(x) \quad \text{for all} \quad c \in T. \tag{9.13}$$

As before, the factor $\chi(c)$ is a character of the algebraic (in our case, torus) group. This fact is known as the Borel–Weil theorem [72]. As such it belongs to the theory of the induced group representations [73].

Remark 9.13. In spite of its apparent simplicity, this theorem is exceptionally deep. It can be used for reconstruction of physical model whose observables obey Eq. (9.13). Examples are given in Refs. [14,74]. Very important contributions to the Borel–Weil–Bott theorem were made recently by Teleman, Refs. [75,76]. These are mainly applicable to the CFT since in his case the loop groups need to be used as explained below. Evidently, models reproducing the Veneziano and Veneziano-like amplitudes as well as all CFT can be reconstructed exclusively with help of the Borel–Weil–Bott theorem as the point of departure.

For physical applications we are also interested in maps Ψ compatible with Eq. (9.2) producing ring of symmetric invariants. Evidently, they are given by

$$\Psi(cx) = \Psi(x). \tag{9.14}$$

To actually obtain these invariants we need to study the orbits of the torus action. To this purpose, in view of Eq. (9.12), we need to consider the following fixed point equation

$$t^a x = x. \tag{9.15}$$

Apart from trivial solutions: $x = 0$ and $x = \infty$, there is a nontrivial solution $t^a = 1$ for any x. For integer a's this is a cyclotomic equation whose nontrivial $a - 1$ solutions all lie on the circle S^1. In view of this circumstance, it is possible to construct the invariants for this case as we would like to explain now. First, such an invariant can be built as a *ratio* of two equivariant mappings of the type given by Eq. (9.13).[29] By construction, such a ratio is the *projective toric* variety. Unlike the affine case, such varieties are *not* represented by functions of homogenous coordinates in $\mathbf{CP}^n$. Instead, they are just constants associated with points in $\mathbf{CP}^n$ which they represent.[30] Second option, is to restrict the algebraic torus to a compact torus. These two options are interrelated in an important way as we would like to explain now.

To this purpose we notice that Eq. (9.12) still holds if some of t-factors are replaced by 1's. This means that one should take into account all situations when one, two, etc. t-factors in Eq. (9.12)

[29] Clearly, the integrand in the period integral, Eq. (3.3), when written in the projective form (as explained in Section 3.1. of Part I) fits this requirement.

[30] This is nontrivial fact. It is used in Ref. [6] and Part III for development of symplectic models reproducing the Veneziano amplitudes.

are replaced by 1's and account for all permutations involving such cases. This observation leads to the torus actions on toric subvarieties. It is very important that different orbits which belong to different subvarieties do not overlap. Thus, by design, X_Σ is the *disjoint union* of *finite* number of orbits identified with the subvarieties of X_Σ. This leads to the flag decomposition, etc. to be discussed further in Parts III and IV. For the sake of space, in this part we consider only the main ideas. For instance, let us consider a specific example of an action of the map Ψ on a monomial $\mathbf{u}^{\mathbf{l}} = u_1^{l_1} \cdots u_n^{l_n} \equiv z_1^{l_1 a_1} \cdots z_n^{l_n 1 a_n}$. For such a map the character $\chi(c)$ is given by

$$\chi(c) = t^{\langle \mathbf{l} \cdot \mathbf{a} \rangle}, \tag{9.16}$$

where $\langle \mathbf{l} \cdot \mathbf{a} \rangle = \sum_i l_i a_i$ and both l_i and a_i are integers. Following Ref. [77], let us consider the limit $t \to 0$ in the above expression. We obtain,

$$c(t) = \begin{cases} 1 & \text{if } \langle \mathbf{l} \cdot \mathbf{a} \rangle = 0 \\ 0 & \text{if } \langle \mathbf{l} \cdot \mathbf{a} \rangle \neq 0. \end{cases} \tag{9.17}$$

The equation $\langle \mathbf{l} \cdot \mathbf{a} \rangle = 0$ describes a hyperplane or, better, a set of hyperplanes for a given vector $\mathbf{a}$. Based on the results of Appendix A, such set forms at least one chamber. To be more accurate, we would like to complicate matters a little bit by introducing a subset $I \subset \{1, \ldots, n\}$ such that, say, only those l_i's which belong to this subset satisfy $\langle \mathbf{l} \cdot \mathbf{a} \rangle = 0$. Naturally, one obtains the one-to-one correspondence between such subsets and earlier mentioned flags. The set of such constructed monomials forms the set of invariants of torus group action. In view of Eq. (9.15), we would like to replace the limiting $t \to 0$ procedure by the limiting procedure requiring $t \to \xi$, where ξ is the nontrivial nth root of unity. After such a replacement we are entering formally the domain of pseudo-reflection groups as explained in Appendix A, part A.4. Such groups are acting on hyperplanes $\langle \mathbf{l} \cdot \mathbf{a} \rangle$. Replacing t by ξ causes us to change the rule, Eq. (9.17), as follows

$$c(\xi) = \begin{cases} 1 \text{ if } \langle \mathbf{l} \cdot \mathbf{a} \rangle = 0 \bmod n \\ 0 \text{ if } \langle \mathbf{l} \cdot \mathbf{a} \rangle \neq 0 \end{cases} \tag{9.18}$$

At this point it is appropriate to recall Eq. (3.11a) of Part I. In view of this equation, we shall call the equation $\langle \mathbf{l} \cdot \mathbf{a} \rangle = n$ as the *Veneziano condition* while the *Kac-Moody-Bloch-Bragg* ($K - M - B - B$) *condition*, Eq. (3.22) of Part I, can be written now as $\langle \mathbf{l} \cdot \mathbf{a} \rangle = 0 \bmod n$.

The results of Appendix A, parts A.3, A.4, indicate that the first option (the Veneziano condition) is characteristic for the standard Weyl–Coxeter (pseudo)reflection groups while the second is characteristic for the *affine* Weyl–Coxeter groups thus leading to the Kac–Moody affine Lie algebras.

Remark 9.14. In Chapter 4, Section 4.4, of Ref. [21]. Ginzburg shows how to recover finite dimensional representations, e.g. $sl_k(\mathbf{C})$, discussed in Section 8, even in the case when K-M-B-B condition is used instead of the Veneziano. In this sense, in accord with Remark 9.13, one can design both CFT and high energy physics observables using the same formalism. Still another option is discussed immediately below.

The arguments just presented provide "physical" proof of the theorem by Serre.

9.2. Additional uses of the theorem by Serre

9.2.1. Connections with the theory of hyperplane arrangements

According to Appendix A, part A.4, both real and complex reflection groups are isometries of respectively $\mathbf{R}^n$ and $\mathbf{C}^n$ spaces leaving some known quadratic forms invariant. The quadratic forms are essentially the hyperplanes. Although Corollary 2.3 to the theorem by Solomon provides a very interesting connection between the hyperplanes and pseudo-reflections, it should be clear that Eqs. (9.18) and (2.6) represent different (but related) sets of hyperplanes (in view of the isomorphism, Eq. (9.4)). The Weyl group W introduced in Appendix A permutes these hyperplanes

Definition 9.15. A collection $\mathcal{A}$ of hyperplanes on which the group W acts transitively is called the *reflection arrangement*.

Remark 9.16. In mathematics literature, one can find a large group of researchers who take the above definition as a starting point of the whole mathematical development presented in this paper, e.g. see Refs. [33,34]. Such an approach is helpful in the following sense. The hypergeometric integrals of the type given by Eq. (3.5) (and those discussed in Part I) can be obtained as solutions of some differential equations (of Picard–Fucs type) whose origin is naturally explained with help of the theory of arrangements. The same equations can be obtained from the point of view of singularity theory, e.g see Ref. [78]. In our earlier work, Ref. [10], we have indeed obtained such type of equations for the Veneziano-type integral, Eq. (3.5), using ideas from singularity theory. Since the theory of arrangements was already applied successfully [79] to reproduce the results of two dimensional CFT [80], it can be used, in principle, as unifying formalism for both "new" string and "old" CFT.

In Appendix A, parts A.3, A.4, we have explained in simple terms the difference between the affine and standard (pseudo)reflection groups. The polyhedra associated with these groups can be thought of as fundamental domains for respective groups of isometries of $\mathbf{R}^n$ and $\mathbf{C}^n$. The action of such isometry groups causes tessellation of these spaces. The situation here is analogous to that encountered in solid state physics [81] (as we mentioned already in Part I) where it is well known that the scattering processes in solids should be treated with account of translational symmetry of the crystal so that the concepts of energy and momentum loose their original meaning and should be modified to account for periodicity.[31] Such situation is characteristic for *all* CFT where one should use the affine Weyl–Coxeter reflection groups and the Kac–Moody algebras associated with them in accord with Proposition A.1 of Appendix A. At the same time, for processes taking place in high energy physics, it is sufficient to consider only the point group symmetries (in solid physics terminology), i.e. the usual Weyl–Coxeter reflection groups and the complete fans associated with them. Such a picture can be extended, if necessary, to include the spherical, hyperbolic and complex hyperbolic spaces so that the polyhedra associated with isometry groups of these spaces will represent the respective fundamental domains. In the case of Kac–Moody algebras such a program was actually implemented (e.g. for the hyperbolic spaces) as described in the book by Kac, Ref. [57]. In the case of high energy physics the next subsection can be used as a point of departure for analogous development.

[31] As a typical illustration we suggest to our readers to think about, is the process of heat or electrical conduction in perfect crystals and its explanation in the solid state physics literature [81].

9.2.2. From complex hyperbolic space to the Heisenberg group

Earlier in Remark 7.4 we have noticed that the Hermitian quadratic form can be extended so that the isometries of the resulting complex space are complex hyperbolic. At that time such a possibility appeared no more than a curiosity. However, upon discussing the exact solution of the Veneziano model in Section 8 we made an observation that the Dirac-type equation associated with the Hamiltonian, Eq. (8.11), is invariant with respect to the Lorentz transformations and it is this invariance which eventually causes connections with zeros of the Riemann zeta function. The connected part of the Lorentz group describes isometries of the hyperbolic space, e.g. see [82, pp. 64–66]. The complex hyperbolic space includes real hyperbolic space as a subspace. According to Goldman, Ref. [47], the pseudo-reflection groups are isometries of the complex hyperbolic space.

In our earlier work, Ref. [83], we discussed various properties of the real hyperbolic space in connection with widely publicized AdS-CFT correspondence. As can be seen either from the book by Thurston, Ref. [82], or from our earlier work, the hyperbolic ball model of the real hyperbolic space is quite adequate for description of many meaningful physical models. In this case, the boundary of the hyperbolic space plays an important role. For instance, the infinitesimal variations at the boundary of the Poincaré disc model – the simplest model of H^2 – produce naturally the Virasoro algebra. Extension of the method producing this algebra to, say, H^3 is complicated by the Mostow rigidity theorem (as discussed in Ref. [83]). This theorem tells us that the Teichmüller space for the hyperbolic three-manifolds without boundaries is a point. Simply speaking, all hyperbolic surfaces without boundaries in hyperbolic space are rigid (nonbendable). This restriction can be lifted in certain cases.

Since the real hyperbolic space is a part of complex hyperbolic and since the real hyperbolic space can be modeled by the hyperbolic ball model, it is not too surprising that the complex hyperbolic space also can be modelled with help of the complex hyperbolic ball model as it is demonstrated in [47]. What is surprising however is that the isometry group at the boundary of this ball model is the Heisenberg group. We would like to argue that to make an extension as suggested in Remark 7.4 is not an artifact but, actually, a necessity. To demonstrate this we need to go to Part I and to take into account the discussion on pages 14 and 15 related to phase factors. This discussion is just an adaptation of results described in Chapter 5 of the monograph by Lang, Ref. [84]. Specifically, on page 77 in connection with calculation of the periods of the Fermat curve, he mentions about a complex plane **C** with two points (0 and 1) deleted. Clearly, the third point is ∞ and, therefore, such trice punctured plane has the hyperbolic disk model as its universal cover. Due to factorisation property of the Veneziano amplitude (e.g. see Eq. (3.28) of Part I), the model of complex projective space discussed earlier in Section 7.6.3 will inherit the hyperbolicity coming from each complex plane **C**. In fact, much more can be said following Ref. [46]. In particular, by analogy with complex sphere (equivalent to a projective space $\mathbf{CP}^1$) with n points removed thus making it hyperbolic one can think about a complex projective space $\mathbf{CP}^n$ with $2n + 1$ hyperplanes removed. In our case, using terminology of hyperplane arrangements, Refs. [33,34], Eq. (2.6) is called a *defining polynomial* for such an arrangement. The entries in such polynomial represent hyperplanes. The polynomial is zero when at least one of its entries becomes zero. In the simplest case of four particle amplitude we have a polynomial Q of the type: $Q = x(1 - x)$ The third hyperplane is at infinity. Clearly, zeros of Q are the same as we just mentioned. The general pattern can be deduced from Eq. (2.8) of Part I. Hence, removal of certain number of hyperplanes from projective space can indeed make it hyperbolic. This is the content of the theorem by Kiernan, Ref. [85], who proved that the manifold $M = \mathbf{CP}^n \setminus \cup_{i=1}^{2n+1} H_i$ is hyperbolic if the hyperplanes H_i are in general position. We would like to prove this result using some physical arguments relevant to calculations of the Veneziano amplitudes. For this purpose it is sufficient only to take another look at our Eq. (1.22) of Part I

$$P(k,t) \equiv \left(\frac{1}{1-t}\right)^{k+1} = \sum_{n=0}^{\infty} p(k,n)t^n. \tag{9.19}$$

As in Eq. (2.8) of Part I, we replace the t-variable in this equation by the variable $t = u_1 + u_1 + \cdots + u_n$. Next, choosing the parametrization of the projective space as described in Section 7.6.3, we would like to consider the generating function $K(\mathbf{z}, \bar{\mathbf{z}})$ of the following type

$$K(\mathbf{z}, \bar{\mathbf{z}}) = \frac{n!}{\pi^n}\left(\frac{1}{1-t}\right)^{n+1}, \tag{9.20}$$

where now $t = \sum_{i=1}^{n} z_i\bar{z}_i \equiv \sum_{i=1}^{n} u_i$ (and the last sum is written as a deformation retract). The constants $n!$ and π^n are chosen in accord with Refs. [46, 47, p. 79]. Such a function is known in geometric function theory, Ref. [46], as the Bergman kernel. It is used as a potential for constriction of (the Bergman) metric by the following rule:

$$\mathrm{d}s^2 = \sum_{i,j=0}^{n} g_{i,j}(\mathbf{z}, \bar{\mathbf{z}})\,\mathrm{d}z_i\,\mathrm{d}\bar{z}_j = \sum_{i,j=0}^{n}\left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\ln K(\mathbf{z}, \bar{\mathbf{z}})\right)\mathrm{d}z_i\,\mathrm{d}\bar{z}_j. \tag{9.21}$$

Also, the same potential is used for construction of the fundamental (1, 1) form which (up to a factor i/π) coincides with the first Chern class [48, pp. 219,220]. Using this observation along with standard facts from the theory of characteristic classes it is rather straightforward now to reproduce the Solomon algebra of invariants. Moreover, since the symplectic manifolds all are of Kähler type and since in the present case the Kähler manifold is of Hodge type, Ref. [48, p. 219], this fact can be used for development of symplectic model reproducing the Veneziano amplitudes. This is done in Ref. [6] and will be further discussed in Part III using more rigorous mathematical arguments. The metric in Eq. (9.21) by design is the metric of the complex hyperbolic ball B^n model. Biholomorphic mappings of B^n are isometries of the Bergman metric, Ref. [47, p. 79]and Ref. [48, p. 219]. *It plays the same role for the complex hyperbolic space as the Lobachevsky metric for the real hyperbolic space*. But, as we know already from our experience with Eqs. (1.19) and (2.8) of Part I, Eq. (9.20) can be identified with the partition (generating) function for the multiparticle Veneziano amplitudes! Thus, we just have demonstrated that inseparable connections between the complex hyperbolic geometry and the Veneziano amplitudes imply the existence of the Heisenberg group at the boundary of B^n. The connections between the hyperbolic geometry inside B^n and the Heisenberg group at the boundary of B^n is explained in detail in Goldman's monograph, Ref. [47]. Our earlier experience with the AdS-CFT correspondence in real hyperbolic space, Ref. [83], suggests that analogous constructions can be made in the complex hyperbolic space. The intrinsic role of the Heisenberg group at the boundary of B^n makes such a project especially attractive.

Thus, the theory of polynomial invariants of finite (pseudo)reflection groups and, especially, Theorem 9.1 by Serre, not only allow us to restore the scattering amplitudes and the generating function associated with them but also impose very rigid constraints on analytical form of such amplitudes thus making them to reflect the symmetries of space-time in which they act. This puts the Veneziano amplitudes into very unique position. Only future might tell if such a position should be replaced by something even more fundamental.

Notes added in proof

1. The results of Sections 8 and 9 can be put in a broader mathematical context with help of the monograph on “Noncommutative Harmonic Analysis” by M.E. Taylor, AMS Publishers, Providence, RI, 1986.
2. Relevance of spin chains to QCD had been discussed by Faddeev and Korchemsky in the paper “High energy QCD as a completely integrable model”, arxiv: hep-th/9404173. The arxiv contains numerous follow up papers based on that, just cited.
3. In Part IV we shall discuss results by Faddeev and Korchemsky using formalism developed in this work.

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Appendix A. Some results from the theory of Weyl–Coxeter reflection and pseudo-reflection groups

A.1. The Weyl group

Let V be a finite dimensional vector space endowed with a scalar product $\langle , \rangle$ which is positive-definite symmetric bilinear form. For each nonzero $\alpha \in V$ let r_α denote the orthogonal reflection in the hyperplane H_α through the origin perpendicular to α. Clearly, the set of hyperplanes H_α is in one-to-one correspondence with the set of α's. For $v \in V$ we obtain,

$$r_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha, \tag{A.1}$$

where $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is the vector *dual* to α. Thus defined reflection is an orthogonal transformation in a sense that $\langle r_\alpha(v), r_\alpha(\mu) \rangle = \langle v, \mu \rangle$. In addition, $[r_\alpha(v)]^2 = 1 \forall \alpha, v$. Conversely, these two properties imply the transformation law, Eq. (A.1). From these results it follows that for $v = \alpha$ we get $r_\alpha(\alpha) = -\alpha$, that is reflection in the hyperplane with change of vector orientation. If the set of vectors $\alpha \in V$ is mutually orthogonal, then $r_\alpha(v) = v$ for $v \neq \alpha$ but, in general, the orthogonality is not required. Because of this, one introduces the *root system* Δ of vectors which span V. Such a system is *crystallographic* if for each pair $\alpha, \beta \in \Delta$ one has

$$\langle \alpha^\vee, \beta \rangle \in \mathbf{Z} \quad \text{and} \quad r_\alpha(\beta) \in \Delta. \tag{A.2}$$

Thus, each reflection $r_\alpha(\alpha \in \Delta)$ permutes Δ. A finite collection of such reflections forms a group W known as the *Weyl group of* Δ. The vectors $\alpha^\vee$ (for $\alpha \in \Delta$) form a root system $\Delta^\vee$ *dual* to Δ. Let $v \in \Delta$ be such that $\langle v, \alpha \rangle \neq 0$ for each $\alpha \in \Delta$. Then, the set Δ^+ of roots $\alpha \in \Delta$ such that $\langle v, \alpha \rangle > 0$ is called a system of *positive* roots of Δ. A root $\alpha \in \Delta^+$ is *simple* if it is not a sum of two elements from Δ^+. The number of simple roots coincides with the dimension of the vector space V and the root set Δ is made of the disjoint union $\Delta = \Delta^+ \amalg \Delta^-$. The integral linear combinations of roots, i.e. $\sum_i m_i\alpha_i$ with m_i' being integers, forms a root *lattice* $Q(\cdot)$ in V (that is free abelian group of rank $n = \dim V$). Clearly, the simple roots form a basis Σ of $Q(\cdot)$. Accordingly, $Q(\cdot^+)$ is made of combinations $\sum_i m_i\alpha_i$ with m_i's being nonnegative integers.

In view of one-to-one correspondence between the set of hyperplanes $\cup_\alpha H_\alpha$ and the set of roots Δ, it is convenient sometimes to introduce the *chambers* as connected components of the complement of $\cup_\alpha H_\alpha$ in V. In the literature, Ref. [17, p. 70], this complement is known also as the *Tits cone*. Accordingly, for a given chamber C_i its *walls* are made of hyperplanes H_α. The roots in Δ can therefore be characterized as those roots which are orthogonal to some wall of C_i and directed towards the interior of this chamber. A *gallery* is a sequence $(C_0, C_1, \ldots, C_l)$ of chambers each of which is adjacent to and distinct from the next. Let $w = r_{i_1} \ldots r_{i_l}$ then, treating the Weyl group W as a chamber system, a gallery from 1 to w can be formally written as $(1, r_{i_1}, r_{i_1}r_{i_2}, \ldots, r_{i_1} \ldots r_{i_l})$. If this gallery is of the shortest possible *length* $l(w)$, then one is saying that $r_{i_1} \ldots r_{i_l}$ is *reduced decomposition* for the word w made of "letters" r_{i_j}. Let C_x and C_y be some distinct chambers which we shall call x and y for brevity. One can introduce the distance function $d(x, y)$ so that, for example, if $w = r_{i_1} \ldots r_{i_l}$ is the reduced decomposition, then $d(x, y) = w$ if and only if there is a gallery of the type $r_{i_1} \ldots r_{i_l}$ from x to y. If, for instance, $d(x, y) = r_i$, this means simply that x and y are distinct and i-adjacent. A *building* $\mathcal{B}$ is a chamber system having a distance function $d(x, y)$ taking values in the Weyl–Coxeter group W. Finally, an *apartment* in a building $\mathcal{B}$ is a subcomplex $\hat{\mathcal{B}}$ of $\mathcal{B}$ which is isomorphic to W. There is a bijection $\varphi : W \to \hat{\mathcal{B}}$ such that $\varphi(w)$ and $\varphi(w')$ are i-adjacent in $\hat{\mathcal{B}}$ if and only if w and w' are adjacent in W, e.g. see Ref. [86].

A.2. The Coxeter group

The Coxeter group is related to the Weyl group through the obviously looking type of relation between reflections,

$$(r_\alpha r_\beta)^{m(\alpha,\beta)} = 1, \tag{A.3}$$

where $m(\alpha, \alpha) = 1$ and $m(\alpha, \beta) \geq 2$ for $\alpha \neq \beta$. In particular, for *finite* Weyl groups $m(\alpha, \beta) \in \{2, 3, 4, 6\}$, Ref. [30, p. 39], while for the *affine* Weyl groups (to be discussed below) $m(\alpha, \beta) \in \{2, 3, 4, 6, \infty\}$, e.g. read Ref. [30, p. 136], and Proposition A.1 below. Clearly, different refection groups will have different matrix $m(\alpha, \beta)$ and, clearly, the matrix $m(\alpha, \beta)$ is connected with the bilinear form (the Cartan matrix, see below) for the Weyl's group W [66,87]. As an example of use of the concept of building in the Weil group, consider the set of *fundamental weights* defined as follows. For the root basis Σ (or $\Sigma^\vee$) the set of fundamental weights $\mathcal{D} = \{\omega_1, \ldots, \omega_n\}$ with respect to Σ is defined by the rule:

$$\langle\alpha_i^\vee, \omega_j\rangle = \delta_{ij}. \tag{A.4}$$

The usefulness of such defined fundamental weights lies in the fact that they allow to introduce the concept of the *highest weight* λ (sometimes also known as the dominant weight, [88, p. 203]. Thus defined λ can be presented as $\lambda = \sum_{i=1}^{d} a_i\omega_j$ with all $a_i \geq 0$. Sometimes it is convenient to relax the definition of fundamental weights to just weights by comparing Eqs. (A.2) and (A.4). That is β's in Eq. (A.2) are just weights. Thus, for instance, we have Δ as building and a subcomplex $\mathcal{D}$ of fundamental weights as an apartment complex.

To illustrate some of these concepts let us consider examples which are intuitively appealing and immediately relevant to the discussion in the main text. These are the root system B_d and C_d. They are made of vector set $\{u_1, \ldots, u_d\}$ constituting an orthonormal basis of the d-dimensional cube. The vectors u_i should not be necessarily of unit length, Ref. [29, p. 27]. It is important only that they all have the same length. For B_d system one normally chooses, Ref. [29, p. 30],

$$\Delta = \{\pm u_i \pm u_j | i \neq j\} \amalg \{\pm u_i\}. \tag{A.5}$$

In this case, the reflections corresponding to elements of Δ can be described by their effect on the set $\{u_1, \ldots, u_d\}$. Specifically, $r_{u_i - u_j} =$ permutation which interchanges u_i and u_j; $r_{u_i} =$ sign change of u_i; $r_{u_i + u_j} =$ permutation which interchanges u_i and u_j and changes their sign. The action of the Weyl group on Δ can be summarized by the following formula

$$W(\Delta) = \left(\frac{\mathbf{Z}}{2\mathbf{Z}}\right)^d \trianglelefteq \Sigma_d, \tag{A.6}$$

with $\trianglelefteq$ representing the semidirect product between the permutation group Σ_d and the dihedral group $(\mathbf{Z}/2\mathbf{Z})^d$ of sign changes both acting on $\{u_1, \ldots, u_d\}$. Thus defined product constitutes the full symmetry group of the d-cube, Ref. [29, p. 31]. The same symmetry information is contained in C_d root system defined by

$$\Delta = \{\pm u_i \pm u_j | i \neq j\} \amalg \{\pm 2u_i\}. \tag{A.7}$$

Both systems possess the same root decomposition: $\Delta = \Delta^+ \amalg \Delta^-$, Ref. [29, p. 37]. In particular, considering a square as an example we obtain the basis Σ_{B_2} of $Q(\cdot)$ as

$$\Sigma_{B_2} = \{u_1 - u_2, u_2\}. \tag{A.8a}$$

From here the dual basis is given by

$$\Sigma_{B_2}^{\vee} = \{u_1 - u_2, 2u_2\}. \tag{A.8b}$$

Using Eq. (A.4) we obtain the fundamental weights as $\omega_1 = u_1$ and $\omega_2 = \frac{1}{2}(u_1 + u_2)$ respectively. By design, they obey the orthogonality condition, Eq. (A.4). The Dynkin diagram, Ref. [29, p. 122], for B_2 provides us with coefficients $a_1 = 1$ and $a_2 = 2$ obtained for the case when the expansion $\lambda = \sum_{i=1}^{d} a_i \omega_j$ is relaxed to $\lambda = \sum_{i=1}^{d} a_i \beta_j$ as discussed above. In view of Eq. (A.8a) this produces at once: $\lambda_{B_2} = u_1 + u_2$. Analogously, for C_2 we obtain,

$$\Sigma_{C_2} = \{u_1 - u_2, 2u_2\}, \tag{A.9}$$

with coefficients $a_1 = 2$ and $a_2 = 2$ thus leading to $\lambda_{C_2} = 2(u_1 + u_2)$.

For the square, these results are intuitively obvious. Evidently, the d-dimensional case can be treated accordingly. The physical significance of the highest weight should become obvious if one compares the Weyl–Coxeter reflection group algebra with that for the angular momentum familiar to physicists. In the last case, the highest weight means simply the largest value of the projection of the angular momentum onto z-axis. The raising operator will annihilate the wave vector for such a quantum state while the lowering operator will produce all eigenvalues lesser than the maximal value (up to the largest negative) and, naturally, all eigenfunctions. The significance of the fundamental weights goes beyond this analogy, however. Indeed, suppose we can expand some root α_i according to the rule

$$\alpha_i = \sum_j m_{ij} \omega_j. \tag{A.10}$$

Then, substitution of such an expansion into Eq. (A.2) and use of Eq. (A.4) produces:

$$\langle \alpha_k^{\vee}, \alpha_i \rangle = \sum_j m_{ij} \langle \alpha_k^{\vee}, \omega_j \rangle = m_{ik}. \tag{A.11}$$

The expression $\langle \alpha_k^{\vee}, \alpha_i \rangle$ is known in the literature as the Cartan matrix. It plays the central role in defining *both* finite and infinite dimensional semisimple Lie algebras [57]. According to Eqs. (A.4), (A.10) and (A.11), the transpose of the Cartan matrix transforms the fundamental weights into the fundamental roots.

A.3. The affine Weyl–Coxeter groups

Physical significance of the affine Weyl–Coxeter reflection groups comes from the following proposition

Proposition A.1. *Let W be the Weyl group of any Kac-Moody algebra. Then W is a Coxeter group for which* $m(\alpha, \beta) \in \{2, 3, 4, 6, \infty\}$. *Any Coxeter group with such* $m(\alpha, \beta)$ *is crystallographic* (*e.g. see* Eq. (A.2))

The proof can be found in Ref. [87, pp. 25–26]. To understand better the affine Weyl–Coxeter groups, following Coxeter, Ref. [89], we would like to explain in simple terms the origin and the physical meaning of these groups. It is being hoped, that such a discussion might significantly facilitate understanding of the results presented in the main text. We begin with the quadratic form

$$\Theta = \sum_{i,j} a_{ij} x_i x_j, \tag{A.12}$$

having symmetric matrix $\|a_{ij}\|$ whose rank is ρ. Such a form is said to be *positive definite* if it is positive for all values of $\mathbf{x} = \{x_1, \ldots, x_n\}$ ($n \geq \rho$ in general !) except zero. It is *positive semidefinite* if it is never negative but vanishes for some x_i's not all zero. The form Θ is indefinite if it can be both positive for some x_i's and negative for others.[32] If the positive semidefinite form vanishes for some $x_i = z_i (i = 1 - n)$, then

$$\sum_i z_i a_{ij} = 0, \qquad j = 1 - n. \tag{A.13}$$

For a given matrix $\|a_{ij}\|$ Eq. (A.13) can be considered as the system of linear algebraic equations for z_i's. Let $\mathcal{N} = n - \rho$ be the *nullity* of the form Θ. Then, it is a simple matter to show that *every positive semidefinite connected* Θ *form is of nullity 1*. The form is connected if it cannot be presented as a sum of two forms involving separate sets of variables. The following two propositions play the key role in causing differences between the infinite affine Weyl–Coxeter (Kac–Moody) algebras and their finite counterparts

Proposition A.2. *For any positive semidefinite connected* Θ *form there exist unique* (*up to multiplication by the common constant*) *positive numbers* z_i *satisfying* Eq. (A.13).

Proposition A.3. *If we modify a positive semidefinite connected* Θ *form by making one of the variables vanish, the obtained form becomes positive definite.*

[32] For the purposes of comparison with existing mathematical physics literature [26] it is sufficient to consider only positive and positive semidefinite forms.

Next, we consider the quadratic form Θ as the norm and the matrix a_{ij} as the metric tensor. Then, as usual, we have $\mathbf{x}\cdot\mathbf{x} = \Theta = |\mathbf{x}|^2$ and, in addition, $\mathbf{x}\cdot\mathbf{y} = \sum_{i,j} a_{ij}x_i y_j \equiv \sum_i x^i y_i = \sum_i x_i y^i$ so that if vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal we get $\sum_{i,j} a_{ij}x_i y_j = 0$ as required. Each vector $\mathbf{x}$ determines a point $(\mathbf{x})$ and a hyperplane $[\mathbf{x}]$ with respect to some reference point $\mathbf{0}$ chosen as an origin. The distance l between a point $(\mathbf{x})$ and a hyperplane $[\mathbf{y}]$ measured along the perpendicular is the projection of $\mathbf{x}$ along the direction of $\mathbf{y}$, i.e.

$$l = \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{y}|}. \tag{A.14}$$

Let now $(\mathbf{x}')$ be the image of $(\mathbf{x})$ by reflection in the hyperplane $[\mathbf{y}]$. Then, $\mathbf{x}-\mathbf{x}'$ is a vector parallel to $\mathbf{y}$ of magnitude $2\,l$. Thus,

$$\mathbf{x}' = \mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{y}|}\mathbf{y}, \tag{A.15}$$

in accord with Eq. (A.1). From here, the equation for the reflecting hyperplane is just $\mathbf{x}\cdot\mathbf{y} = 0$. Let the vector $\mathbf{y}$ be pre assigned, then taking into account Propositions A.2 and A.3 we conclude that for the nullity $\mathcal{N} = 0$ the only solution possible is $\mathbf{x} = 0$. That is to say, in such a case n reflecting hyperplanes have the point $\mathbf{0}$ as the only common intersection point. A complement of these hyperplanes in $\mathbf{R}^n$ forms a chamber system discussed already in A.1). In the main text it is called a complete fan in accordance with existing terminology [24,25]. For $\mathcal{N} = 1$ the equation $\mathbf{x}\cdot\mathbf{y} = 0$ may have many *nonnegative* solutions for $\mathbf{x}$. Actually, such reflecting hyperplanes occur in a finite number of different directions. More accurately, such hyperplanes belong to a *finite* number of families, each consisting of hyperplanes *parallel to each other*. If we choose a single representative from each family in a such a way that it passes through $\mathbf{0}$, then the complement of such representatives is going to form a polyhedral cone as before. But now, in addition, we have a group of translations T for each representative of the hyperplane family so that the total affine Weyl group W_{aff} is the semidirect product: $W_{\text{aff}} = T \trianglelefteq W$. The fundamental region for W_{aff} is a simplex (to be precise, an open simplex, Bourbaki, Ref. [17], Chapter 5, Proposition 10) called *alcove* bounded by $n+1$ hyperplanes (walls) n of which are reflecting hyperplanes passing through $\mathbf{0}$ while the remaining one serves to reflect $\mathbf{0}$ into another point $\mathbf{0}'$. If one connects $\mathbf{0}$ with $\mathbf{0}'$ and reflects this line in other hyperplanes one obtains a lattice. By analogy with solid state physics [81] one can construct a dual lattice (just like in A.1 and A.2 above) the fundamental cell of which is known in physics as the Brilluin zone. For the alcove the fundamental region of the dual lattice (the Brilluin zone) is the polytope having $\mathbf{0}$ for its centre of symmetry, i.e. zonotope [89].

A.4. The pseudo-reflection groups

Although the pseudo-reflection groups are also formally described in the monograph by Bourbaki, Ref. [17], their geometrical (and potentially physical) meaning is beautifully explained in the book by McMullen [32]. In particular, all earlier presented reflection groups are isometries of the Euclidean space. Their action preserves some quadratic form which is real. More generally, one can think of reflections in spherical and hyperbolic spaces. From this point of view earlier described polytopes (polyhedra) represent the fundamental regions for respective isometry groups. Action of these groups on fundamental regions causes tesselation of these spaces (without gaps). The collection of spaces can be enlarged by considering reflections in the complex n-dimensional space $\mathbf{C}^n$. In this case the Euclidean quadratic form is replaced by the positive definite Hermitian form. Since locally $\mathbf{CP}^n$ is the same as $\mathbf{C}^{n+1}$ and since $\mathbf{CP}^n$ is at the same time a symplectic manifold with well known symplectic two-form Ω [27], this makes the pseudo-reflection groups (which leave Ω invariant) especially attractive for physical applications (e.g. see Section 9 and Part III). The pseudo-reflections are easily described. By analogy with Eq. (A.1) (or (A.15)) one writes

$$r_\alpha(v) = v + (\xi - 1)\langle v, \alpha^\vee \rangle \alpha, \tag{A.16}$$

where ξ is the nontrivial solution of the cyclotomic equation $x^h = x$ and $\alpha^\vee = \alpha/\langle \alpha, \alpha \rangle$ with $\langle x, y \rangle$ being a positive definite Hermitian form satisfying as before $\langle r_\alpha(v), r_\alpha(\mu) \rangle = \langle v, \mu \rangle$ with α being an eigenvector such that $r_\alpha(\alpha) = \xi \alpha$.[33] In addition, $[r_\alpha(\alpha)]^k = \xi^k \alpha$ for $1 \le k \le h - 1$. This follows from the fact that

$$[r_\alpha(\nu)]^k = \nu + (1 + \xi + \cdots + \xi^{k-1})(\xi - 1)\langle v, \alpha^\vee \rangle \alpha, \tag{A.17}$$

and taking into account that $(1 + \xi + \cdots + \xi^{k-1})(\xi - 1) = \xi^k - 1$.

Finally, the Weyl–Coxeter reflection groups considered earlier in this Appendix can be treated as pseudo-reflection groups if one replaces a single Euclidean reflection by the so called Coxeter element [19,29] ω which is a product of individual reflections belonging to the distinct roots of Δ. Hence, the Euclidean Weyl–Coxeter reflection groups can be considered as a subset of the pseudo-reflection groups so that all useful information about these groups can be obtained from considering the same problems for the pseudo-reflection groups. It can be shown [19,29] that the Coxeter element ω has eigenvalues $\xi^{m_1}, \ldots, \xi^{m_l}$ with l being dimension of the vector space Δ while the exponents $m_1, \ldots, m_l$ being positive integers less than h and such that $\sum_{i=1}^{l}(h - m_i) = \sum_{i=1}^{l} m_i$. This result implies that the number $\sum_{i=1}^{l} m_i = N$—the number of positive roots in the Weyl–Coxeter group is connected with the Coxeter number h via relation: $N = \frac{1}{2} lh$, Ref. [30, p. 79].

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